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FRACTIONAL DERIVATIVE PERTAINING TO I-FUNCTION AND LAURICELLA FUNCTION

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Abstract :
The object of this paper is to obtain a fractional derivative of I- function associated with generalize Lauricella functions and general class of multivariable polynomials.

Key words: Fractional derivative operator, I-function, Lauricella function, general class of multivariable polynomials.

1. INTRODUCTION:

The I- function given by Saxena [4] is represented and defined as following:

$$I^{e,f}_{p_i,q_i;R}[Z]=I^{e,f}_{p_i,q_i;R}\left[z\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,f},(a_{ji},\alpha_{ji})_{f+1,p_i}\\(b_j,\beta_j)_{1,e},(b_{ji},\beta_{ji})_{e+1,q_i}\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int \phi(\xi)\,z^{\xi}\,d\xi \qquad \ldots (1.1)$$

Where

$$\phi(\xi)=\frac{\prod_{j=1}^{e}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{f}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{R}c_i\prod_{j=f+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\prod_{j=e+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)} \qquad \ldots (1.2)$$

p_i (i= 1,….,R), q_i (i= 1,….,R), e, f are integers satisfying $0 \le f \le p_i$, $0 \le e \le q_i$ (i= 1,….,R); R is finite, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$, are real and positive; a_j, b_j, a_{ji}, b_{ji} are complex numbers and £ is the path of integration separating the increasing and decreasing sequences of poles of the integrand.

The integral converges if

$$|\arg(z)|<\frac{\pi}{2}\Omega_i,$$

Where $$\Omega_i=\sum_{j=1}^{f}\alpha_j-\sum_{j=f+1}^{p_i}\alpha_j+\sum_{j=1}^{e}\beta_j-\sum_{j=e+1}^{q_i}\beta_j>0$$

And

$$T=\sum_{j=1}^{q_i}b_j-\sum_{j=1}^{p_i}a_j>0 \qquad \ldots (1.3)$$

If we take R=1 in (1.1), then the I- function will convert to the well known Fox's H- function.

The general class of polynomials (multivariable) defined by Srivastava and Garg [7] represented in the following manner:

$$S_{f}^{e_1,\dots,e_r}[x_1,\dots,x_r]=\sum_{k_1,\dots\dots k_r=0}^{e_1k_1+\dots+e_rk_r\le f}\frac{(-f)_{e_1k_1+\dots\dots+e_rk_r}}{k_1!\dots\dots k_r!}\,.\,A[f\,;k_1;\dots;k_r]\,x_1^{k_1}\dots\dots x_r^{k_r}\qquad \dots (1.4)$$

Where $e_{1,\dots,}e_r = 0$, are arbitrary positive integers and the coefficients $A[f\,;k_1;\dots;k_r]$ are arbitrary constants, real or complex.

The fractional derivatives of a function f (t) of complex order γ given by Oldham and Spanner [3] and Srivastava and Goyal [8] as following:

$$ {}_0D_t^{\gamma}\,\{f(t)\} = \begin{cases} \dfrac{1}{\Gamma(-\gamma)}\int_0^t (t-x)^{-\gamma-1}\,f(x)\,dx & ; \quad Re(\gamma)<0\,, \\ \dfrac{d^m}{dt^m}\,D_t^{\gamma-m}\,\{f(t)\} & ; \quad 0\le \mathrm{Re}(\gamma)<m \end{cases} \qquad \dots (1.5)$$

Where m is a positive integer

The special case of the fractional derivatives of Oldham and Spanier [3] is

$$D_t^{\gamma}\,(t^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\gamma+1)}\,t^{\mu-\gamma}\quad ,\quad \mathrm{Re}(\mu)>-1 \qquad \dots (1.6)$$

The following known result is given by Srivastava and Panda [9]

Lemma:

If λ (≥0), 0 <x<1, Re (1+p) > 0 , Re (q) > -1, $\lambda_i > 0$ and $\Delta_i > 0$ or $\Delta_i = 0$ and $|z_i| < \sigma$, i= 1,2,…,r then

$$x^{\lambda}\,.\,F\begin{bmatrix} z_1x^{\lambda_1} \\ \vdots \\ z_rx^{\lambda_r}\end{bmatrix} = \sum_{m=0}^{\infty}\frac{(1+p+q+2M)\,(-\lambda)_M\,(1+p)_{\lambda}}{M!\quad (1+p+q+M)_{\lambda+1}}.\ F_M\,[z_1,\dots,z_r]\ .\ {}_2F_1\left(\begin{matrix}-M,1+p+q+M\ ; \\ 1+p\ ;\end{matrix}\ x\right) \qquad \dots (1.7)$$

(For M ≥0)

Where $F_M\,[z_1,\dots,z_r]$

$$= F^{E+2:\,U^{(1)};\dots\dots;U^{(r)}}_{P+2:\,V^{(1)};\dots\dots;V^{(r)}}\left(\begin{matrix}[(e):\eta^{(1)};\dots\dots;\eta^{(r)}],\ [1+p+\lambda:\ \lambda_1;\dots;\lambda_r],[\lambda+1;\dots;\lambda_r] & : [(w^1):x^1];\dots;[w^{(r)}:x^{(r)}] \\ [(g):\xi^1;\dots\dots;\xi^{(r)}],[2+p+q+M+\lambda:\lambda_1;\dots;\lambda_r],\ [\lambda-M+1;\lambda_1;\dots;\lambda_r] & : [(V^1):t^1];\dots\dots;[V^{(r)}:t^{(r)}]\end{matrix}\ ;\ z_1,\dots,z_r\right) \qquad \dots (1.8)$$

Now following shorthand notations given by Srivastava and Daoust [6] denote the generalized Lauricella function of several complex variables.

$$F^{E:\,U^{(1)};\dots\dots;U^{(r)}}_{P:\,V^{(1)};\dots\dots;V^{(r)}}\quad[\gamma_1,\dots,\gamma_r] = F\,[\gamma_1,\dots,\gamma_r] \qquad \dots (1.9)$$

2. THE MAIN RESULT :

This result consists I- function, Lauricella function and general class of polynomials as following:

$$D_x^{\gamma}\left\{\begin{array}{c}(x-u)^{\alpha}\,\zeta^{\alpha}\,(v-x)^{\alpha+\beta}\,F\begin{pmatrix}\eta_1\{\zeta\,(v-x)\}^{\alpha_1}\\ \vdots\\ \eta_s\{\zeta\,(v-x)\}^{\alpha_s}\end{pmatrix}\;S_f^{e_1,\ldots,e_r}\begin{pmatrix}(x-u)^{c_1}\,(v-x)^{d_1}\\ \vdots\\ (x-u)^{c_r}\,(v-x)^{d_r}\end{pmatrix}\\ I_{p_i,q_i;R}^{e,f}\left[t\{x(x-u)\}^{\alpha^1}\{x(v-x)\}^{\beta^1}\left|\begin{array}{l}(c_j,\sigma_j)_{1,f},(c_{ji},\sigma_{ji})_{f+1,p_i}\\ (d_j,\rho_j)_{1,e},(d_{ji},\rho_{ji})_{e+1,q_i}\end{array}\right.\right]\end{array}\right\}$$

$$=\sum_{M=0}^{\infty}\;\sum_{k_1,\ldots\ldots,k_r=0}^{e_1k_1+\ldots.+e_rk_r\le f}\;\sum_{m_1=0}^{\infty}\;\sum_{n_1=0}^{\infty}\;\sum_{h=0}^{\infty}\frac{(-f)_{e_1k_1+\ldots\ldots+e_rk_r}}{k_1!\ldots\ldots k_r!}.\Delta.\,A[f;k_1;\ldots;k_r]$$

$$I_{p_i+3,q_i+3;R}^{e,f+3}\left[t(-u)^{\alpha^1}(v)^{\beta^1}x^{\alpha^1+\beta^1}\left|\begin{array}{l}\left[(-m_1-n_1),\alpha^1+\beta^1\right],\left[-\alpha-\sum_{i=1}^{r}c_ik_i;\alpha^1\right],\left[-\beta-h-\sum_{i=1}^{r}d_ik_i;\beta^1\right],(c_j,\sigma_j)_{1,f};(c_{ji},\sigma_{ji})_{f+1,p_i}\\ (d_j,\rho_j)_{1,e},\left[m_1-\alpha-\sum_{i=1}^{r}c_ik_i;\alpha^1\right],\left[n_1-\beta-h-\sum_{i=1}^{r}d_ik_i;\beta^1\right],\left[(\gamma-m_1-n_1);\alpha^1+\beta^1\right];(d_{ji},\rho_{ji})_{e+1,q_i}\end{array}\right.\right]$$

…(2.1)

Where

$$\Delta=(-1)^{m_1}\frac{(1+p+q+2M)\,(-\alpha_m)\,(-M)_h\,(1+p)_{\alpha}(1+p+q+M)_h}{M!\;(1+p+q+M)_{\alpha+1}\,(1+p)_h\,\Gamma(m_1+1)\,\Gamma(n_1+1)}\,\zeta^h(-u)^{\alpha-m_1+\sum_{i=1}^{r}c_ik_i}\,(v)^{\beta+h+\sum_{i=1}^{r}d_ik_i}$$

$$.(x)^{m_1+n_1-\gamma}\,F_M(\delta_1,\ldots,\delta_s);$$

$\alpha_i>0$ $\beta_i>0$, i= 1, 2…, s ; and

$$\mathrm{Re}\,(\alpha)+\sum_{i=1}^{s}\alpha_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1,$$

$$\mathrm{Re}\,(\beta)+\sum_{i=1}^{s}\beta_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1 \qquad \ldots(2.2)$$

Proof: In order to prove (2.1), express the I-function in terms of Mellin-Barnes type of contour integrals by (1.1) and Lauricella function by (1.7) and general class of polynomials given by (1.4), then collecting the powers of (x-u) and (v-x). Finally making use of the result (1.6), we get the main result (2.1).

3. PARTICULAR CASES:

(I) If we take R=1 in (1.1), then I-function breaks into well known Fox's H-function and consequently there hold the following result:

$$D_x^{\gamma}\left\{\begin{array}{c}(x-u)^{\alpha}\,\zeta^{\alpha}\,(v-x)^{\alpha+\beta}\,F\begin{pmatrix}\eta_1\{\zeta(v-x)\}^{\alpha_1}\\ \vdots\\ \eta_s\{\zeta(v-x)\}^{\alpha_s}\end{pmatrix}\,S_f^{e_1,\dots,e_r}\begin{pmatrix}(x-u)^{c_1}(v-x)^{d_1}\\ \vdots\\ (x-u)^{c_r}(v-x)^{d_r}\end{pmatrix}\\ H_{p,q}^{e,f}\left[t\{x(x-u)\}^{\alpha^1}\{x(v-x)\}^{\beta^1}\left|\begin{array}{l}(c_j,\sigma_j)_{1,p}\\(d_j,\rho_j)_{1,q}\end{array}\right.\right]\end{array}\right\}$$

$$=\sum_{M=0}^{\infty}\sum_{k_1,\dots,k_r=0}^{e_1k_1+\dots+e_rk_r\le f}\sum_{m_1=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-f)_{e_1k_1+\dots+e_rk_r}}{k_1!\dots k_r!}\,\Delta.\,A[f;k_1;\dots;k_r]$$

$$H_{p+3,q+3}^{e,f+3}\left[t(-u)^{\alpha^1}(v)^{\beta^1}x^{\alpha^1+\beta^1}\left|\begin{array}{l}\left[(-m_1-n_1),\alpha^1+\beta^1\right],\left[-\alpha-\sum_{i=1}^{r}c_ik_i;\alpha^1\right],\left[-\beta-h-\sum_{i=1}^{r}d_ik_i;\beta^1\right],(c_j,\sigma_j)_{1,p}\\(d_j,\rho_j)_{1,q},\left[m_1-\alpha-\sum_{i=1}^{r}c_ik_i;\alpha^1\right],\left[n_1-\beta-h-\sum_{i=1}^{r}d_ik_i;\beta^1\right],\left[(\gamma-m_1-n_1);\alpha^1+\beta^1\right]\end{array}\right.\right] \quad \dots (3.1)$$

Valid under the conditions surrounding (2.1)

(I) If we take multivariable H-function in place of I-function in (2.1), then we have a known result obtained by Chaurasia and Singhal [2] as following:

$$D_x^{\gamma}\left\{(x-u)^{\alpha}\,\zeta^{\alpha}\,(v-x)^{\alpha+\beta}\,F\begin{pmatrix}\eta_1\{\zeta(v-x)\}^{\alpha_1}\\ \vdots\\ \eta_s\{\zeta(v-x)\}^{\alpha_s}\end{pmatrix}\,S_f^{e_1,\dots,e_r}\begin{pmatrix}(x-u)^{c_1}(v-x)^{d_1}\\ \vdots\\ (x-u)^{c_r}(v-x)^{d_r}\end{pmatrix}\,H\left[\begin{array}{c}t_1\\ \vdots\\ t_s\end{array}\left|\begin{array}{c}\{x(x-u)\}^{\alpha_1}\{x(v-x)\}^{\beta_1}\\ \\ \{x(x-u)\}^{\alpha_s}\{x(v-x)\}^{\beta_s}\end{array}\right.\right]\right\}$$

$$=\sum_{M=0}^{\infty}\sum_{k_1,\dots,k_r=0}^{e_1k_1+\dots+e_rk_r\le f}\sum_{m_1=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-f)_{e_1k_1+\dots+e_rk_r}}{k_1!\dots k_r!}\,\Delta.\,A[f;k_1;\dots;k_r]$$

$$.\,H_{A+3,C+3;[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,\lambda+3:(x',y');\dots;(x^{(r)},y^{(r)})}\left[\begin{array}{c}t_1(-u)^{\alpha_1}.v^{\beta_1}.x^{\alpha_1+\beta_1}\\ \vdots\\ t_s(-u)^{\alpha_s}.v^{\beta_s}.x^{\alpha_s+\beta_s}\end{array}\left|\begin{array}{l}[-m_1-n_1;\alpha_1+\beta_1:\dots:\alpha_s+\beta_s],\left[-\alpha-\sum_{i=1}^{r}c_ik_i;\alpha_1;\dots;\alpha_s\right],\left[-\beta-h-\sum_{i=1}^{r}d_ik_i;\beta_1;\dots;\beta_s\right],\\ \left[(c):\Psi';\dots;\Psi^{(s)}\right],\left[m_1-\alpha-\sum_{i=1}^{r}c_ik_i;\alpha_1;\dots;\alpha_s\right],\left[n_1-\beta-h-\sum_{i=1}^{r}d_ik_i;\beta_1;\dots;\beta_s\right],\end{array}\right.\right.$$

$$\left.\begin{matrix} [(a):\theta^{(1)}:\ldots:\theta^{(s)}]:[b^{(1)}:\phi^{(1)}];\ldots;[b^{(s)}:\phi^{(s)}] \\ \\ \left[(\gamma-m_1-n_1);\alpha_1+\beta_1:\ldots\ldots:\alpha_s+\beta_s\right]:[(d'):\delta'];\ldots;[(d^{(s)}):\delta^{(s)}] \end{matrix}\right] \quad \ldots (3.2)$$

Valid under the conditions surrounding (2.1)

(II) Taking R=1 and replacing $f \to f_1, \ldots, f_r$ in (2.1), we get a known result obtained by Chaurasia and Shekhawat [1].

4. Conclusion:

The main result derived here is of a very general Nature and hence encompass several cases of interest hitherto scattered in the literature.

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