FRACTIONAL DERIVATIVE PERTAINING TO I-FUNCTION AND LAURICELLA FUNCTION

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Abstract:
The object of this paper is to obtain a fractional derivative of I-function associated with generalize Lauricella functions and general class of multivariable polynomials.

Key words: Fractional derivative operator, I-function, Lauricella function, general class of multivariable polynomials.

1. INTRODUCTION:

The I-function given by Saxena [4] is represented and defined as following:

\[ I_{\begin{array}{c} a_{1} \\ \vdots \\ a_{R} \end{array}}^{e,f} \left[ \begin{array}{c} b_{1} \\ \vdots \\ b_{R} \end{array} \right] \left( z \right) = \frac{1}{2\pi i} \int_{\mathbf{L}} \phi(\zeta) z^{\zeta} d\zeta \]

where

\[ \phi(\zeta) = \prod_{j=1}^{e} \Gamma(\beta_{j} - \alpha_{j} \zeta) \prod_{j=1}^{f} \Gamma(1 - \alpha_{j} + \alpha_{j} \zeta) \]

\[ \sum_{i=1}^{R} c_{i} \prod_{j=f+1}^{p_{i}} \Gamma(\alpha_{ji} - \alpha_{ji} \zeta) \prod_{j=e+1}^{q_{i}} \Gamma(1 - \beta_{ji} + \beta_{ji} \zeta) \]

\[ p_{i} (i=1, \ldots, R), q_{i} (i=1, \ldots, R), e, f \] are integers satisfying \( 0 \leq f \leq p_{i}, 0 \leq e \leq q_{i} \) (\( i=1, \ldots, R \)); \( R \) is finite, \( \alpha_{j}, \beta_{j}, a_{ji}, b_{ji} \) are real and positive; \( a_{j}, b_{j}, a_{ji}, b_{ji} \) are complex numbers and \( \mathbf{L} \) is the path of integration separating the increasing and decreasing sequences of poles of the integrand.

The integral converges if

\[ |\arg(z)| < \frac{\pi}{2} \Omega_{i} \]

where

\[ \Omega_{i} = \sum_{j=1}^{f} \alpha_{j} - \sum_{j=f+1}^{q_{i}} \alpha_{j} + \sum_{j=1}^{e} \beta_{j} - \sum_{j=e+1}^{q_{i}} \beta_{j} > 0 \]

And

\[ T = \sum_{j=1}^{q_{i}} b_{j} - \sum_{j=1}^{p_{i}} a_{j} > 0 \]
If we take \( R=1 \) in (1.1), then the \( I \)-function will convert to the well-known Fox's H-function.

The general class of polynomials (multivariable) defined by Srivastava and Garg [7] represented in the following manner:

\[
S_{t_1,\ldots,t_r}^{f_{e_1},\ldots,e_r} [x_1,\ldots,x_r] = \sum_{k_1,\ldots,k_r=0}^c \frac{(-f)^{e_1+k_1+\ldots+e_r+k_r}}{k_1!\ldots k_r!} A[f; k_1;\ldots; k_r] x_1^{k_1}\ldots x_r^{k_r} \quad \ldots (1.4)
\]

Where \( e_1,\ldots,e_r = 0 \), are arbitrary positive integers and the coefficients \( A[f; k_1;\ldots; k_r] \) are arbitrary constants, real or complex.

The fractional derivatives of a function \( f(t) \) of complex order \( \gamma \) given by Oldham and Spanier [3] and Srivastava and Goyal [8] as following:

\[
\begin{align*}
0^D_t \gamma \{ f(t) \} & = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(-\gamma)} \int_0^t (t-x)^{-\gamma-1} f(x) \, dx & ; \quad Re(\gamma) < 0, \\
\frac{d^m}{dt^m} D_t^\gamma \{ f(t) \} & ; \quad 0 \leq Re(\gamma) < m
\end{array} \right. \ldots (1.5)
\end{align*}
\]

Where \( m \) is a positive integer

The special case of the fractional derivatives of Oldham and Spanier [3] is

\[
D_t^\mu (t^\mu) = \frac{\Gamma(\mu+1)}{\Gamma(-\mu+1)} t^{\mu-\gamma} , \quad Re(\mu) > -1 \quad \ldots (1.6)
\]

The following known result is given by Srivastava and Panda [9]

**Lemma:**

If \( \lambda \geq 0 \), \( 0 < x < 1 \), \( Re(1+p) > 0 \), \( Re(q) > -1 \), \( \lambda_i > 0 \) and \( \Delta_i > 0 \) or \( \Delta_i = 0 \) and \( |z_i| < \sigma \), \( i=1,2,\ldots,r \) then

\[
x^\lambda \cdot F \left[ \begin{array}{c}
z_1 x^{\lambda_1} \\
\vdots \\
z_r x^{\lambda_r}
\end{array} \right] = \sum_{m=0}^\infty \frac{(1+p+q+2M)(-\lambda)M(1+p)\lambda}{M! (1+p+q+M)_{\lambda+1}} \cdot F_M [z_1,\ldots,z_r] \cdot 2F_1 \left[ \begin{array}{c}
-M,1+p+q+M \\
1+p
\end{array} \right] \ldots (1.7)
\]

(For \( M \geq 0 \))

Where \( F_M [z_1,\ldots,z_r] \)

\[
= \frac{p^{E+2}: U^{(1)};\ldots;U^{(r)}}{p+2: V^{(1)};\ldots;V^{(r)}}
\]

\[
\left[ \begin{array}{c}
(\epsilon; \eta^{(1)};\ldots;\eta^{(r)}), [1+p+\lambda;\lambda_1;\ldots;\lambda_r], [\lambda+1;\ldots;\lambda_r] \\
(\xi; \zeta^{(1)};\ldots;\zeta^{(r)}), [2+p+q+M+\lambda;\lambda_1;\ldots;\lambda_r], [\lambda-M+1;\lambda_1;\ldots;\lambda_r]
\end{array} \right] = \left[ \begin{array}{c}
(\omega^{(1)}; x^{(1)};\ldots;w^{(r)}; x^{(r)}), [V^{(1)}; t^{(1)};\ldots;V^{(r)}; t^{(r)}] \\
\{z_1,\ldots,z_r\}
\end{array} \right] \ldots (1.8)
\]

Now following shorthand notations given by Srivastava and Daoust [6] denote the generalized Lauricella function of several complex variables.

\[
P_{E; U^{(1)};\ldots,U^{(r)}} [Y_1,\ldots,Y_r] = F [Y_1,\ldots,Y_r] \quad \ldots (1.9)
\]
2. THE MAIN RESULT:

This result consists of function, Lauricella function and general class of polynomials as following:

\[
D_x^\gamma \left\{ \int_{p_i q_i, R} \left[ t \left\{ x(x - u)^{\beta_1} (v - x)^{\alpha_1 + \beta} \right\} \right] \right\}
\]

\[
= \sum_{M=0}^{\infty} \sum_{k_i} \sum_{r_i} \sum_{h=0}^{\infty} \frac{(-f)^{r_i} e_k e_r}{k_i! \cdots k_r!} \cdot A[f; k_1; \ldots; k_i]
\]

...(2.1)

Where

\[
\Delta = \left( -1 \right)^{m_1} \frac{1 + p + q + 2M}{M!} \cdot \frac{(-M_h)(1+p)(1+q+M)_h}{(1+p+q+M)_{h+1} \Gamma(m_1+1) \Gamma(n_1+1)} \cdot \frac{\zeta_{\alpha}^h(-u)}{(v)^{\alpha_1 + \beta}}
\]

\[
= \frac{m^{m_1 + n_1 - \gamma}}{F_m(-\delta_1, \ldots, -\delta_{\gamma})};
\]

\[\alpha_i > 0, \beta_i > 0, i = 1, 2, \ldots, s; \text{ and} \]

\[\text{Re}(\alpha) + \sum_{i=1}^{s} \alpha_i \left( \frac{d_j}{\delta_j^{(i)}} \right) > -1, \]

\[\text{Re}(\beta) + \sum_{i=1}^{s} \beta_i \left( \frac{d_j}{\delta_j^{(i)}} \right) > -1 \quad \text{...(2.2)} \]

**Proof:** In order to prove (2.1), express the I-function in terms of Mellin-Barnes type of contour integrals by (1.1) and Lauricella function by (1.7) and general class of polynomials given by (1.4), then collecting the powers of \((x-u)\) and \((v-x)\). Finally making use of the result (1.6), we get the main result (2.1).
3. PARTICULAR CASES:

(I) If we take R=1 in (1.1), then I-function breaks into well known Fox’s H-function and consequently there hold the following result:

\[
D^y_x \left\{ (x-u)^a \zeta^a (v-x)^{a+\beta} F \left( \eta_1 (\zeta(v-x))^{a_1} \ldots \eta_x (\zeta(v-x))^{a_x} \right) S^{\alpha_1, \ldots, \alpha_x}_{r_1, \ldots, r_x} \right\} 
= \sum_{M=0}^{\infty} \sum_{k_1, \ldots, k_M=0}^{c} \sum_{m_1=0}^{\infty} \sum_{n_1=0}^{\infty} \frac{(-f)^{c_1, \ldots, c_M}}{k_1! \ldots k_M!} \Delta \left[ f; k_1; \ldots; k_r \right]
\]

\[
H^A_{p+3,q+3} \left[ t(-u)^a (v)\beta^a (x)\alpha^a \right] \left\{ \begin{array}{l}
\left[-m_1 - n_1, \alpha + \beta, \ldots; \alpha, \beta \right] \\
\left[-\alpha - \sum_{i=1}^{k} c_i, \alpha, \ldots; \alpha \right] \\
\left[-\beta - h - \sum_{i=1}^{k} d_i, \beta, \ldots; \beta \right]
\end{array} \right\} \Delta A \left[ f; k_1; \ldots; k_r \right]
\]

Valid under the conditions surrounding (2.1)

(II) If we take multivariable H-function in place of I-function in (2.1), then we have a known result obtained by Chaurasia and Singhal [2] as following:

\[
D^y_x \left\{ (x-u)^a \zeta^a (v-x)^{a+\beta} F \left( \eta_1 (\zeta(v-x))^{a_1} \ldots \eta_x (\zeta(v-x))^{a_x} \right) S^{\alpha_1, \ldots, \alpha_x}_{r_1, \ldots, r_x} \right\} 
= \sum_{M=0}^{\infty} \sum_{k_1, \ldots, k_M=0}^{c} \sum_{m_1=0}^{\infty} \sum_{n_1=0}^{\infty} \frac{(-f)^{c_1, \ldots, c_M}}{k_1! \ldots k_M!} \Delta \left[ f; k_1; \ldots; k_r \right]
\]

\[
H^A_{p+3,q+3} \left[ t(-u)^a (v)\beta^a (x)\alpha^a \right] \left\{ \begin{array}{l}
\left[-m_1 - n_1, \alpha + \beta, \ldots; \alpha, \beta \right] \\
\left[-\alpha - \sum_{i=1}^{k} c_i, \alpha, \ldots; \alpha \right] \\
\left[-\beta - h - \sum_{i=1}^{k} d_i, \beta, \ldots; \beta \right]
\end{array} \right\} \Delta A \left[ f; k_1; \ldots; k_r \right]
\]
\[(a) : \theta^{(i)} ; \ldots ; \theta^{(s)}] : [b^{(i)} : \phi^{(i)}] ; \ldots ; [b^{(s)} : \phi^{(s)}] \]

\[\left[(\gamma - m_{l} - n_{l}) ; \alpha_{l} + \beta_{l} ; \ldots ; \alpha_{s} + \beta_{s} ; [(d') : \delta'] ; \ldots ; [(d^{(s)}) : \delta^{(s)}] \right] \quad \ldots (3.2)\]

Valid under the conditions surrounding (2.1)

(II) Taking \(R=1\) and replacing \(f \rightarrow f_{1}, \ldots , f_{r}\) in (2.1),
we get a known result obtained by Chaurasia and Shekhawat [1].

4. Conclusion:
The main result derived here is of a very general
Nature and hence encompass several cases of interest
hitherto scattered in the literature.

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