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APPLICATIONS OF LIE GROUP IN TWO - DIMENSIONAL HEAT EQUATION

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ABSTRACT

In the present paper we obtained the most general solution of two-dimension conduction of heat in a finite rod having the constant thermal diffusivity k_0 by using the general prolongation formula for their symmetry. Several authors obtained solution of heat equation using different type of method [3], [4], [5], [8], [15], [16]. In recent year the authors Horak and Gruber [3], Kurt [4] worked for finding the solution of two-dimension heat equation.

Key Words: Scaling, Translation, Linearity, Galilean-Boost.

1. Introduction:

1.1 The General Prolongation formula:

Let

$$v = \sum_{i=1}^{p} \xi^i (x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha (x,u) \frac{\partial}{\partial u^\alpha}$$

be a vector field defined on an open subset M ⊂ X × U where X is the space of independent variables, U is the space of dependent variables, p is the number of independent variables and q is the number of dependent variables for the system. Then n^{th}-prolongation of v is the vector field

$$pr^{(n)}v = v + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J \left(x, u^{(n)}\right) \frac{\partial}{\partial u_J^\alpha}$$

defined on the corresponding space $M^{(n)} \subset X \times U^{(n)}$ where X is the space of the independent variables, $U^{(n)}$ is the space of the dependent variables and the derivative of the dependent variables up-to n (order of differential equation). The second summation being over all unordered multi-indices $J = (j_1,\dots,j_k)$ with $1 \le j_k \le p$, $1 \le k \le n$. The coefficient function ϕ_α^J of $pr^{(n)} v$ are given by the following formula (1.1.1)

$$\phi_\alpha^J \left(x, u^{(n)}\right) = D_J \left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha$$

where $u_i^\alpha = \left(\partial u^\alpha / \partial x^i\right)$ and $u_{J,i}^\alpha = \left(\partial u_J^\alpha / \partial x^i\right)$ [7].

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1.2 Theorem:

Suppose $\Delta_d(x, u^{(n)})$, for $d = 1,\ldots,l$ is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M and

$$pr^{(n)}v\,[\Delta_d\,(x, u^{(n)})] = 0$$

for $d = 1,\ldots,l$, whenever $\Delta(x, u^{(n)}) = 0$ for every infinitesimal generator v of G, then G is a symmetry group of the system, [7].

2. Solution of the Heat conduction equation:

In this section, we obtained the most general solution by calculate the symmetries for two – dimensional (2-D) Heat conduction in finite rod, without source, with the following assumptions (a). The position of the rod is in the space, (b). The rod is homogeneous, (c). The Heat is uniformly distributed over its cross sectional area at a given time t, (d). The surface of the rod is insulated to prevent any loss of Heat through the boundary, (e). $u(x, y, t)$ is the temperature at the point (x, y) at time t, (f). k_0 be the constant thermal diffusivity, and then the equation to the problem of 2-D Heat conduction in a rod is

$$u_t = k_0\,(u_{xx} + u_{yy}) \qquad (2.1)$$

which is the second order differential equation with three independent variables and one dependent variable, so in our notation p = 3, n = 2 and q =1, [1], [9].

Let (1.2.1)

$$v = \xi(x, y, t, u)\partial_x + \eta(x, y, t, u)\partial_y + \tau(x, y, t, u)\partial_t + \phi(x, y, t, u)\partial_u \qquad (2.2)$$

be a vector field on X×U. Now, determined the second prolongation of v by using (1.1.2)

$$pr^{(2)}v = +\phi^x\partial_{u_x} + \phi^y\partial_{u_y} + \phi^t\partial_{u_t} + \phi^{xx}\partial_{u_{xx}} + \phi^{xy}\partial_{u_{xy}} + \phi^{xt}\partial_{u_{xt}} + \phi^{yy}\partial_{u_{yy}} + \phi^{yt}\partial_{u_{yt}} + \phi^{tt}\partial_{u_{tt}}$$

The coefficients present in (2.3) are calculated by using (1.1.3), applied $pr^{(2)}v$ to (2.1), the infinitesimal criterion (1.2.1) takes the form

$$\phi^t = k_0\,(\phi^{xx} + \phi^{yy}) \qquad (2.4)$$

Substituted the value of ϕ^t, ϕ^{xx} and ϕ^{yy} in equation (2.4) and replaced u_t by $k_0\,(u_{xx} + u_{yy})$ whenever it present in equation (2.4) and then equated the coefficients of the terms in the first and second order partial derivatives of u, the determining equations for the symmetry group of (2.1) are found as follow.

Table 1: The Determine Equation Table

Mono-mial	Coefficient	Eq. No.	Mono-mial	Coefficient	Eq. No.
1	$\phi^t = k_0\,(\phi^{xx} + \phi^{yy})$	(1)	$u_y\,u_{yt}$	$-2\,k_0\,\tau_u = 0$	(16)
u_x	$-\xi_t = k_0\,(2\phi_{xu} - \xi_{xx} - \xi_{yy})$	(2)	u_{xx}	$k_0(\phi_u - \tau_t) = k_0\,((\phi_u - 2\xi_x) - k_0\,(\tau_{xx} + \tau_{yy}))$	(17)
u_y	$-\eta_t = k_0\,(2\phi_{yu} - \eta_{xx} - \eta_{yy})$	(3)	u_{yy}	$k_0(\phi_u - \tau_t) = k_0\,((\phi_u - 2\eta_y) - k_0\,(\tau_{xx} + \tau_{yy}))$	(18)
u_x^2	$0 = k_0\,(\phi_{uu} - 2\xi_{xu})$	(4)	u_{xx}^2	$-k_0^2\,\tau_u = -k_0^2\,\tau_u$	(19)

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u_y^2	$0 = k_0(\phi_{uu} - 2\eta_{yu})$	(5)	u_{yy}^2	$-k_0^2\tau_u = -k_0^2\tau_u$	(20)
u_x^3	$0 = -k_0\xi_{uu}$	(6)	$u_{xx}u_{yy}$	$-2k_0^2\tau_u = -2k_0^2\tau_u$	(21)
u_y^3	$0 = -k_0\eta_{uu}$	(7)	$u_x^2 u_y$	$0 = -k_0\eta_{uu}$	(22)
$u_x u_y$	$0 = -k_0(2\eta_{xu} + 2\xi_{yu})$	(8)	$u_x^2 u_{xx}$	$0 = -k_0^2\tau_{uu}$	(23)
$u_x u_{xx}$	$-k_0\xi_u = -k_0(2k_0\tau_{xu} + 3\xi_u)$	(9)	$u_x^2 u_{yy}$	$0 = -k_0^2\tau_{uu}$	(24)
$u_x u_{yy}$	$-k_0\xi_u = -(2k_0\tau_{xu} + \xi_u)$	(10)	$u_y^2 u_x$	$0 = -k_0\xi_{uu}$	(25)
$u_y u_{xx}$	$-k_0\eta_u = -k_0(2k_0\tau_{yu} + \eta_u)$	(11)	$u_y^2 u_{xx}$	$0 = -k_0^2\tau_{uu}$	(26)
$u_y u_{yy}$	$-k_0\eta_u = -k_0(2k_0\tau_{yu} + 3\eta_u)$	(12)	$u_y^2 u_{yy}$	$0 = -k_0^2\tau_{uu}$	(27)
$u_x u_{xy}$	$-2k_0\eta_u = 0$	(13)	u_{xy}	$0 = -k_0(2\eta_x + 2\xi_y)$	(28)
$u_y u_{xy}$	$-2k_0\xi_u = 0$	(14)	u_{xt}	$0 = -2k_0\tau_x$	(29)
$u_x u_{xt}$	$-2k_0\tau_u = 0$	(15)	u_{yt}	$0 = -2k_0\tau_y$	(30)

The requirement for (15), (16), (29) and (30) is that τ be a function of t. The equations (9), (10), (11), (12), (13) and (14) shown that ξ and η are independent of u. From equation (17) and (18) $\xi = (1/2)x\,\tau_t + \sigma(y, t)$ and $\eta = (1/2)y\,\tau_t + \gamma(x, t)$ where σ and γ are arbitrary functions. The equation (4) and (5) shown that ϕ is linear in u so $\phi(x, y, t, u) = \beta(x, y, t)\,u + \alpha(x, y, t)$ for certain function α and β. The equation (2) and (3) gives $\beta_x = -(1/2\,k_0)\xi_t$ and $\beta_y = -(1/2\,k_0)\eta_t$ which is implies $\beta = -(1/8\,k_0)\,(x^2 + y^2)\,\tau_{tt} - (1/2\,k_0)(x\,\sigma_t + y\,\gamma_t) + \rho(t)$ where ρ is only function of t. The equation (1) requires that both α and β are the solutions of (2.1), i.e., $\alpha_t = k_0(\alpha_{xx} + \alpha_{yy})$ and $\beta_t = k_0(\beta_{xx} + \beta_{yy})$. Using the determining equation of β we found that $-(1/8\,k_0)(x^2 + y^2)\,\tau_{ttt} - (1/2\,k_0)(x\,\sigma_{tt} + y\,\gamma_{tt}) + \rho_t = -(1/2\,k_0)\tau_{tt}$ which gives us $\tau = c_1 + c_6\,2t + c_7\,4t^2$, $\sigma = c_8\,2t + \theta(y)$, $\gamma = c_9\,2t + \psi(x)$ and $\rho = -c_7\,4t + c_4$. The equation (28) gives $\sigma_y = \gamma_x$ which implies $\psi = c_3 - c_5\,x$ and $\theta = c_2 + c_5\,y$. Since all the determining equations are satisfied. The most general infinitesimal symmetry of (2.1) has coefficient functions of the form $\xi = c_2 + c_6\,x + c_5\,y + c_8\,2t + c_7\,4xt$, $\eta = c_3 - c_5\,x + c_6\,y + c_9\,2t + c_7\,4yt$, $\tau = c_1 + c_6\,2t + c_7\,4t^2$ and $\phi = [c_4 - (1/k_0)\{(c_8\,x + c_9\,y) + c_7\,(x^2 + y^2 + 4k_0\,t)\}]\,u + □(x, y, t)$ where $c_1,\ldots,c_9$ are arbitrary constants and $□(x, y, t)$ is an arbitrary solution of (2.1). The Lie algebras of infinitesimal symmetries of (2.1) is spanned by the nine vector field $v_1 = \partial_t$, $v_2 = \partial_x$, $v_3 = \partial_y$, $v_4 = u\partial_u$, $v_5 = y\partial_x - x\partial_y$, $v_6 = x\partial_x + y\partial_y + 2t\partial_t$, $v_7 = 4t\,(x\partial_x + y\partial_y) + 2t^2\,\partial_t - ((x^2 + y^2 + 4k_0\,t)/k_0)\,u\partial_u$, $v_8 = 2t\,\partial_x + (-1/k_0)\,xu\partial_u$, $v_9 = 2t\,\partial_y + (-1/k_0)\,yu\partial_u$, and the infinite-dimensional sub algebra $v_{□} = □(x, y, t)\partial_u$, where □ is an arbitrary solution of (2.1).

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The commutation table gives the commutation relations between these vector fields as follows.

Table 2: The Commutation Relation Table

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_α
v_1	0	0	0	0	0	$2v_1$	$4(v_6-v_4)$	$2v_2$	$2v_3$	v_{α_t}
v_2	0	0	0	0	$-v_3$	v_2	$2v_8$	$-v_4/k_0$	0	v_{α_x}
v_3	0	0	0	0	v_2	v_3	$2v_9$	0	$-v_4/k_0$	v_{α_y}
v_4	0	0	0	0	0	0	0	0	0	$-v_\alpha$
v_5	0	v_3	$-v_2$	0	0	0	0	v_9	$-v_8$	v_{α_1}
v_6	$-2v_1$	$-v_2$	$-v_3$	0	0	0	$2v_7$	v_8	v_9	v_{α_2}
v_7	$-4(v_6-v_4)$	$-2v_8$	$-2v_9$	0	0	$-2v_7$	0	0	0	v_{α_3}
v_8	$-2v_2$	v_4/k_0	0	0	$-v_9$	$-v_8$	0	0	0	v_{α_4}
v_9	$-2v_3$	0	v_4/k_0	0	$-v_8$	$-v_9$	0	0	0	v_{α_5}
v_α	$-v_{\alpha_t}$	$-v_{\alpha_x}$	$-v_{\alpha_y}$	v_α	$-v_{\alpha_1}$	$-v_{\alpha_2}$	$-v_{\alpha_3}$	$-v_{\alpha_4}$	$-v_{\alpha_5}$	0

Where $\alpha_1 = y\alpha_x - x\alpha_y$, $\alpha_2 = x\alpha_x + y\alpha_y + 2t\alpha_t$, $\alpha_3 = 4t\,(x\alpha_x + y\alpha_y + t\alpha_t)$, $\alpha_4 = (2t\alpha_x + \alpha(x/k_0))$, $\alpha_5 = (2t\alpha_y + \alpha(y/k_0))$. The one - parameter groups G_i generated by the v_i are given as follows

G_1: $(x, y, t + \varepsilon, u)$, G_2: $(x + \varepsilon, y, t, u)$, G_3: $(x, y + \varepsilon, t, u)$, G_4: $(x, y, t, e^{\varepsilon} u)$,
G_5: $(x \cos\varepsilon + y \sin\varepsilon, y \cos\varepsilon - x \sin\varepsilon, t, u)$, G_6: $(e^{\varepsilon} x, e^{\varepsilon} y, e^{2\varepsilon} t, u)$,
G_7: $\left(\frac{x}{1-4\varepsilon t}, \frac{y}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon t}, u \cdot \sqrt{(1-4\varepsilon t)} \cdot \exp\left(\frac{-\varepsilon\left(x^2+y^2\right)}{k_0\left(1-4\varepsilon t\right)}\right)\right)$
G_8: $(x + 2\varepsilon t, y, t, u.exp(-(\varepsilon x + \varepsilon^2 t) / k_0))$, G_9: $(x, y + 2\varepsilon t, t, u.exp(-(\varepsilon y + \varepsilon^2 t) / k_0))$,
G_α: $(x, y, t, u + \varepsilon\, \alpha(x, y, t))$ where each group G_i is a symmetry group.

If we take $u = f(x, y, t)$ be a solution of (2.1), then the functions

$u^{(1)} = f(x, y, t - \varepsilon)$, $u^{(2)} = f(x - \varepsilon, y, t)$, $u^{(3)} = f(x, y - \varepsilon, t)$, $u^{(4)} = e^{\varepsilon} f(x, y, t)$, $u^{(5)} = f(x \cos\varepsilon - y \sin\varepsilon, y \cos\varepsilon + x \sin\varepsilon, t)$, $u^{(6)} = f(e^{-\varepsilon} x, e^{-\varepsilon} y, e^{-2\varepsilon} t)$, $u^{(7)} = \frac{1}{\sqrt{(1+4\varepsilon t)}} \exp\left(\frac{-\varepsilon\left(x^2+y^2\right)}{k_0\left(1+4\varepsilon t\right)}\right) f\left(\frac{x}{1+4\varepsilon t}, \frac{y}{1+4\varepsilon t}, \frac{t}{1+4\varepsilon t}\right)$

$u^{(8)} = exp((-\varepsilon x + \varepsilon^2 t) / k_0).\, f(x - 2\varepsilon t, y, t)$, $u^{(9)} = exp((-\varepsilon y + \varepsilon^2 t) / k_0).\, f(x, y - 2\varepsilon t, t)$, $u^{(\alpha)} = f(x, y, t) + \varepsilon\, \alpha(x, y, t)$, are any other solutions of (2.1), where ε is any real number and $\alpha(x, y, t)$ any other solution to (2.1). At the end the most general solution of (2.1) obtained from a given solution $u = f(x, y, t)$, by group transformations is in the form given below

$$u=\frac{1}{\sqrt{(1+4\varepsilon_7 t)}} \exp\left(\varepsilon_4 - \frac{\left(\left(\varepsilon_8 x-\varepsilon_8^2 t\right)/k_0\right)+\left(\left(\varepsilon_9 y-\varepsilon_9^2 t\right)/k_0\right)+\varepsilon_7 x^2+\varepsilon_7 y^2}{k_0\left(1+4\varepsilon_7 t\right)}\right)$$

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$$f\left(\frac{e^{-\varepsilon_6}\left(x\cos\varepsilon_5-2\varepsilon_8 t\right)}{1+4\varepsilon_7 t}-y\sin\varepsilon_5-\varepsilon_2,\ \frac{e^{-\varepsilon_6}\left(y\cos\varepsilon_5-2\varepsilon_9 t\right)}{1+4\varepsilon_7 t}+x\sin\varepsilon_5-\varepsilon_3,\ \frac{e^{-2\varepsilon_6}t}{1+4\varepsilon_7 t}-\varepsilon_1,\right)+\alpha(x,y,t) \qquad (2.5)$$

where $\varepsilon_1,\ldots,\varepsilon_9$ are real constants and α an arbitrary solution of (2.1).

3. Conclusion:

In our investigation the symmetry groups G_4 and G_α reflect the linearity of (2.1). The groups G_2, G_3 and G_1 are the time and space invariance of the equation respectively, and reflect the fact that (2.1) has constant coefficients. The group G_6 is well known scaling symmetry group. The group G_8 and G_9 represent a kind of Galilean boost to a moving coordinate frame. The group G_5 is well known rotational symmetry group. The group G_7 is a genuinely local group of transformations and if u = c be a constant solution then the function

$$u = (c/\sqrt{1+4\varepsilon t})\cdot exp(-\varepsilon(x^2+y^2)/k_0(1+4\varepsilon t))$$

be a solution. The most general solution (2.5) gives us all possible most general infinitesimal symmetry of (2.1). The fundamental solution of (2.1) be obtained by substituting $c=\sqrt{\varepsilon/\pi}$, at the point (x_0, y_0, t_0) = (0, 0, (-1/4ε)). Now, by translating the above solution in t using G_1, with ε replaced by -1/4ε, we get the fundamental solution of the problem in the form $u = (1/\sqrt{4\pi t})\cdot exp(-(x^2+y^2)/4k_0 t)$.

4. Special Cases:

(1) Letting y → 0, in (2.1), we get a known result of [2] in the form

$$u=\frac{1}{\sqrt{\left(1+4\varepsilon_7 t\right)}}\exp\left(\varepsilon_4-\frac{\left(\left(\varepsilon_8 x-\varepsilon_8^2 t\right)/k_0\right)+\varepsilon_7 x^2}{k_0\left(1+4\varepsilon_7 t\right)}\right)$$

$$f\left(\frac{e^{-\varepsilon_6}\left(x-2\varepsilon_8 t\right)}{1+4\varepsilon_7 t}-\varepsilon_2,\ \frac{e^{-2\varepsilon_6}t}{1+4\varepsilon_7 t}-\varepsilon_1,\right)+\alpha(x,t) \qquad (4.1)$$

which is a solution of the heat conduction equation in one dimension having the constant thermal diffusivity k_0 , where $\varepsilon_1,\ldots,\varepsilon_9$ are real constants and α an arbitrary solution to the Heat equation.

(2) Letting y → 0 and k_0 = 1 in (2.1), we get a known result of [7] in the form

$$u=\frac{1}{\sqrt{\left(1+4\varepsilon_7 t\right)}}\exp\left(\varepsilon_4-\frac{\varepsilon_8 x-\varepsilon_8^2 t+\varepsilon_7 x^2}{\left(1+4\varepsilon_7 t\right)}\right)$$

$$f\left(\frac{e^{-\varepsilon_6}\left(x-2\varepsilon_8 t\right)}{1+4\varepsilon_7 t}-\varepsilon_2,\ \frac{e^{-2\varepsilon_6}t}{1+4\varepsilon_7 t}-\varepsilon_1,\right)+\alpha(x,t) \qquad (4.2)$$

which is a solution of the heat conduction equation in one dimension, where $\varepsilon_1,\ldots,\varepsilon_9$ are real constants and α an arbitrary solution to the Heat equation.

References:

1. Churchill, R. V., "Operational Mathematics 3rd edition, McGraw-Hill Kogakusha Ltd., 1972.

2. Gupta, V.G. and Pal, Kapil, Application of Symmetry Groups in Heat Equation (Abstracts), Paper Presented at 12th Annual Conference and National Symposium on Applications of Special Functions, Vijnana Parishad of India, JNVU, Jodhpur, Rajasthan, Oct. 25-27, 2007, 45-46.

3. Horak, V. and Gruber, P., Parallel numerical solution of 2-D heat equation, 47-56

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http://www.cosy.sbg.ac.at/events/parnum05/book/horak1.pdf

4. Kurt, N., Solution of two-dimensional heat equation for a square in terms of elliptic functions, Journal of the franklin Institute, vol 345, issue 3, 2008, pp. 303-317.

5. Lebelo, R. S., Fedotov, I. And Shatolov, M., Solution of heat equation with variable coefficient using derive, Buttelspoort time, peer-reviewed. Conference proceeding, 2008.

6. Olver, P. J., Symmetry Groups and Group Invariant Solutions of Partial Differential Equation, J. Diff. Geom. 14, 1979, 497-542.

7. Olver, P. J., Applications of Lie Groups to Differential Equations, Second Edition, Springer-Verlag, New York, 1993.

8. Richards, D., and Abrahamsen, A., The heat equation with partial differential equation, http://online.redwoods.cc.ca.us/instruct/darnold/DEProj/spo2/AbeRichards/slideshowdefinal.pdf

9. Simmons, G. F., "Differential Equations with Applications and Historical Notes", 2nd ed., Mc. Graw Hill, Inc., New York, 1991.

10. Warner, F. W., Foundations of differentiable Manifolds and Lie Groups, Scott., Foresman, Glenview, Ill., 1971

11. Widder, D.V., "The Heat Equation", Academic Press, New York, 1975.

12. The two-dimensional heat equation, http://www.math.utah.edu/classes/216/assignment_09.html

13. Heat equation_wikipedia, the free encyclopedia, http://en.wikipedia.org/wiki/heat_equation

14. Fundamental solution_wikipedia, the free encyclopedia, http://en.wikipedia.org/wiki/fundamental_solution

15. S.M. Moawad, "Variational principles for ideal MHD of steady incompressible flows via Lie-point symmetries with application to the magnetic structures of bipolar sunspots", *The European Physical Journal Plus*, vol. 135, no. 585, pp.1-22, 2020, doi:10.1140/epjp/s13360-020-00598-z.

16. M.B. Abd-el-Malek, N.A. Badran, A.M. Amin and A.M. Hanafy, "Lie symmetry group for unsteady free convection boundary-layer flow over a vertical surface," *Symmetry*, vol.13, no. 175, 2021, doi:10.3390/sym13020175.

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