Available online at https://www.gyanvihar.org/researchjournals/ctm_journals.php
SGVU International Journal of Convergence of Technology and Management
E-ISSN: 2455-7528
Vol.8 Issue 1 Page No 47-50

WAVE PROPAGATION FOR VIBRATING UNIFORM MEMBRANCE

Ashok Singh Shekhawat[1], Kapil Pal[2] and Vatsala Pawar[2]
[1]Jaipur Engineering College and Research Centre, Jaipur, Rajasthan, India
[2]Jaipur National University, Jaipur, Rajasthan, India.

(2000 AMS Mathematics Subject Classification Number: 17B66, 22E70, 35Q75, 46N50, 70G65, 81R05)

ABSTRACT

In the present paper we obtain most general solution for the two dimensional displacement of a vibrating uniform membrane, by computing the symmetry groups using the general prolongation formula for their infinitesimal generators of a groups of transformations based on the technique given by Olver([1], [2]) in explicit form.

Key Word: Vibrating Uniform Membrane, Space Invariance, Translation, Rotation, Commutation -Relation.

1. Introduction:

1.1 About Helmholtz Equation:

Helmholtz type equation arose naturally in many physical applications related to wave propagation, vibration phenomena and heat transfer. These equations are often used to describe the vibration of a structure, the acoustic cavity problem, the radiation wave, the scattering of a wave, heat conduction in fins and acoustic scattering in fluid solid problems. The displacement of a vibrating membrane uxx + uyy = - k2 u (1.1.1) Where k depends on the surface tension and density of the membrane Watson [3]. G. Bluman et. al. [5, 6] studied the partial differential equation system and their symmetries. R. O. Popovych et. al. [7, 8] described the variational symmetries and conservation law of the wave propagation.

1.2 The General Prolongation formula:

Let

$$v=\sum_{i=1}^{p}\xi^{i}\left(x,u\right)\frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{q}\phi_{\alpha}\left(x,u\right)\frac{\partial}{\partial u^{\alpha}} \quad (1.2.1)$$

be a vector field defined on an open subset $M \subset X \times U$ where X is the space of independent variables, and U is the space of dependent variables, p is the number of independent variables and q is the number of dependent variables for the system. Then n^{th}-prolongation of v is the vector field

$$pr^{(n)}v=v+\sum_{\alpha=1}^{q}\sum_{J}\phi_{\alpha}^{J}\left(x,u^{(n)}\right)\frac{\partial}{\partial u_{J}^{\alpha}} \quad (1.2.2)$$

Correspondence to: Ashok Singh Shekhawat, [1]Jaipur Engineering College and Research Centre, Jaipur, Rajasthan, India.
Corresponding author. E-mail addresses: ashoksinghshekhawat.maths@jecrc.ac.in

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$ where X is the space of the independent variables, $U^{(n)}$ is the dependent variables and the derivative of the dependent variables up-to n (order of differential equation) Olver [2].

2. Main Result:

We find the most general solution by calculating the symmetries for two-dimensional Helmholtz equation for the displacement of a vibrating uniform membrane. In equation (1.1.1),

$u_{xx} + u_{yy} = - k^2 u$ (2.1)

which is the second order differential equation with two independent variables and one dependent variable, so in our notation p = 2, n = 2 and q = 1. A vector field on X×U takes the form $v = \xi (x, y, u)\, \partial_x + \eta (x, y, u)\, \partial_y + \phi (x, y, u)\, \partial_u$ (2.2) where ξ, η and ϕ are the smooth coefficient functions. Using (1.2.2) to determine the second prolongation of v,

$$pr^{(2)}v = v + \phi^x \partial_{u_x} + \phi^y \partial_{u_y} + \phi^{xx} \partial_{u_{xx}} + \phi^{xy} \partial_{u_{xy}} + \phi^{yy} \partial_{u_{yy}}$$

(2.3) and the coefficients present in (2.3) can be calculated by using (1.2.3). In case of the two-dimensional Helmholtz equation, by using infinitesimal criterion of invariance equation (1.4.2) takes the form $\phi^{xx} + \phi^{yy} + k^2 \phi = Q(u_{xx} + u_{yy} + k^2 u)$ (2.4) in which $Q(x, y, u^{(2)})$ depend up-to second order derivatives of u. By substituting the values of ϕ^{xx}, ϕ^{yy} and ϕ in equation (2.4) and equating the coefficients of the terms in the first and second order partial derivatives of u, the determining equations for the symmetry group of the two-dimensional Helmholtz equation are found as follows

Monomial	Coefficient	Equation Number
u_{xx}	$\phi_u - 2\xi_x = Q$	(1)
u_{yy}	$\phi_u - 2\eta_y = Q$	(2)
u_{xy}	$-2\eta_x - 2\xi_y = 0$	(3)
$u_x u_{xx}$	$-3\xi_u = 0$	(4)
$u_x u_{yy}$	$-\xi_u = 0$	(5)
$u_x u_{xy}$	$-2\eta_u = 0$	(6)
$u_y u_{xx}$	$-\eta_u = 0$	(7)
$u_y u_{yy}$	$-3\eta_u = 0$	(8)
$u_y u_{xy}$	$-2\xi_u = 0$	(9)
u_x^2	$\phi_{uu} - 2\xi_{xu} = 0$	(10)
$u_x u_y$	$-2\xi_{yu} - 2\eta_{xu} = 0$	(11)
u_y^2	$\phi_{uu} - 2\eta_{yu} = 0$	(12)
u_x	$2\phi_{xu} - \xi_{xx} - \xi_{yy} = 0$	(13)
u_y	$2\phi_{yu} - \eta_{xx} - \eta_{yy} = 0$	(14)
1	$\phi_{xx} + \phi_{yy} + k^2\phi - k^2 Q\, u = 0$	(15)
u_x^3	$-\xi_{uu} = 0$	(16)
u_y^3	$-\eta_{uu} = 0$	(17)
$u_y^2 u_x$	$-\xi_{uu} = 0$	(18)
$u_x^2 u_y$	$-\eta_{uu} = 0$	(19)

Correspondence to: Ashok Singh Shekhawat, [1]Jaipur Engineering College and Research Centre, Jaipur, Rajasthan, India.
Corresponding author. E-mail addresses: ashoksinghshekhawat.maths@jecrc.ac.in

The requirement for equation (4) to (9) is that ξ and η are independent of u, equation (1) and (2) gives $\xi_x = \eta_y$, equation (3) gives $\eta_x = -\xi_y$, equation (10) and (12) gives $\phi = \beta u + \alpha$ where $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ are functions. Form the equation (13) and (14) we get $\beta_x = 0$, $\beta_y = 0$, from (15) we find $\beta = Q = (c_3 / \mathrm{k}^2)$. The most general infinitesimal symmetry of the two-dimensional Helmholtz equation for the displacement of a vibrating uniform membrane has coefficient function of the form $\xi = c_4 y + c_1$, $\eta = - c_4 x + c_2$ and $\phi = (c_3 / \mathrm{k}^2) u + \alpha$ where $c_1,\dots,c_4$ are arbitrary constant and α is an arbitrary solution of the Helmholtz equation. The Lie algebras of infinitesimal symmetries of two-dimensional Helmholtz equation for the displacement of a vibrating uniform membrane is spanned by the four vector fields $v_1 = \partial_x$, $v_2 = \partial_y$, $v_3 = (u / \mathrm{k}^2)\partial_u$, $v_4 = y\partial_x - x\partial_y$ and the infinite-dimensional sub-algebra $v_\alpha = \alpha\partial_u$ where α is an arbitrary solution of two-dimensional Helmholtz equation. The commutation relation between these vector fields are given by the following commutation-relation table

$$\begin{array}{cccccc}
 & v_1 & v_2 & v_3 & v_4 & v_\alpha \\
v_1 & 0 & 0 & 0 & v_2 & v_{\alpha_x} \\
v_2 & 0 & 0 & 0 & v_1 & v_{\alpha_y} \\
v_3 & 0 & 0 & 0 & 0 & -\left(v_\alpha / \mathrm{k}^2\right) \\
v_4 & -v_2 & -v_1 & 0 & 0 & v_{\alpha_1} \\
v_\alpha & -v_{\alpha_x} & -v_{\alpha_y} & \left(v_\alpha / \mathrm{k}^2\right) & v_{\alpha_1} & 0
\end{array}$$

(Where $\alpha_1 = y\,\alpha_x - x\,\alpha_y$).

The one-parameter groups G_i generated by the v_i are given as follows G_1: $(x + \varepsilon, y, u)$, G_2: $(x, y + \varepsilon, u)$, G_3: $(x, y, e^{(\varepsilon/\mathrm{k}^2)} u)$, G_5: $(x\cos\varepsilon + y\sin\varepsilon, y\cos\varepsilon - x\sin\varepsilon, u)$, G_α: $(x, y, u + \varepsilon\alpha)$ where each G_i is a symmetry group. If we take $u = f(x, y)$ be a solution of the Helmholtz equation then are the functions $u^{(1)} = f(x - \varepsilon, y)$, $u^{(2)} = f(x, y - \varepsilon)$, $u^{(3)} = e^{(\varepsilon/\mathrm{k}^2)} f(x, y)$, $u^{(4)} = f(x\cos\varepsilon - y\sin\varepsilon, y\cos\varepsilon + x\sin\varepsilon)$, $u^{(\alpha)} = f(x, y) + \varepsilon\alpha(x, y)$ where ε is any real number and α any other solution to two dimensional Helmholtz equation.

3. Conclusion:

In this investigation, the established result is very useful in many interesting situations appearing in the literature on mathematical analysis, applied chemistry and mathematical physics with the help of our result. In our investigation the symmetry group G_3 and G_α reflects the linearity of two-dimensional Helmholtz equation for the displacement of a vibrating uniform membrane. The group G_1 and G_2 are space translation symmetry group. The group G_4 represent rotational symmetry group. At the end the most general solution that we can obtain from a given solution $u = f(x, y)$, by group transformations is in the form given below

$$u = e^{(\varepsilon_3/\mathrm{k}^2)} f\left(x\cos\varepsilon_4 - y\sin\varepsilon_4 - \varepsilon_1, y\cos\varepsilon_4 + x\sin\varepsilon_4 - \varepsilon_2\right) + \alpha(x, y) \quad (3.1)$$

Correspondence to: Ashok Singh Shekhawat, [1]Jaipur Engineering College and Research Centre, Jaipur, Rajasthan, India.
Corresponding author. E-mail addresses: ashoksinghshekhawat.maths@jecrc.ac.in

where $\varepsilon_1,\ldots,\varepsilon_4$ are real constant and α be an arbitrary solution to two-dimensional Helmholtz equation for the displacement of a vibrating uniform membrane.

4. Special Cases

4.1 If we take k = 1 then equation 3.1 reduces to

$$u = e^{\varepsilon_3} f\left(x\cos\varepsilon_4 - y\sin\varepsilon_4 - \varepsilon_1, y\cos\varepsilon_4 + x\sin\varepsilon_4 - \varepsilon_2\right) + \alpha(x,y) \quad (4.1)$$

where $\varepsilon_1,\ldots,\varepsilon_4$ are real constant and α be an arbitrary solution to two-dimensional Helmholtz equation for the displacement of a vibrating uniform membrane.

4.2 If we take y = 0 and k = 1 then equation 3.1 reduces to

$$u = e^{\varepsilon_3} f\left(x - \varepsilon_1\right) + \alpha\left(x\right) \qquad (4.2)$$

where $\varepsilon_1,\ldots,\varepsilon_4$ are real constant and α be an arbitrary solution to two-dimensional Helmholtz equation for the displacement of a vibrating uniform membrane.

References:

1. Olver, P. J., "Symmetry Groups and Group Invariant Solutions of Partial Differential Equation", J. Diff. Geom. 14, 497-542,1979.

2. Olver, P. J., "Application of Lie Groups to Differential Equations", Second Edition, Springer - Verlag, New York, Inc., 1993.

3. R C., et al., "Applications of the complete multiple reciprocity method for solving the Helmholtz equation of a semi-infinite domain", Advances in Engineering Software 32, 111-117, 2001.

4. Watson, W. W., "The Physical Principles of Wave Guide Transmission and Antenna System", Oxford at the Clarendon Press, 1947.

5. G. Bluman, A.F. Cheviakov, N.M. Ivanova, Framework for nonlocally related partial differential equation systems and nonlocal symmetries: Extension, simplification, and examples, J. Math. Phys. 47 (2006) 113505.

6. G.W. Bluman, A.F. Cheviakov, S.C. Anco, Application of Symmetry Methods to Partial Differential Equations, Springer, New York, 2010.

7. R.O. Popovych, A. Bihlo, Inverse problem on conservation laws, Physica D 401 (2020) 132175, arXiv:1705.03547.

8. Roman O. Popovych, Alexei F. Cheviakov, "Variational symmetries and conservation laws of the wave equation in one space dimension" Applied Mathematics Letters, Volume 104, June 2020, 106225, DOI: 10.1016/j.aml.2020.106225.

Correspondence to: Ashok Singh Shekhawat, [1]Jaipur Engineering College and Research Centre, Jaipur, Rajasthan, India.
Corresponding author. E-mail addresses: ashoksinghshekhawat.maths@jecrc.ac.in