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# *Review article* A REVIEW PAPER ON FRACTIONAL CALCULUS WITH THEIR APPLICATIONS Dr. Govind Sharma<sup>\*</sup>, Himani Aggrawal

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# KeywordsAbstractFractional CalculusThe main object of this paper is to present a brief elementary and<br/>introductory overview of the theory of the integral and derivative<br/>operators of fractional calculus and their applications especially in<br/>developing solutions of certain interesting families of ordinary and<br/>partial fractional "differ integral" equations and used several<br/>important definitions for fractional-order derivatives, including: the<br/>Riemann-Liouville, the Grünwald-Letnikov, the Liouville-Caputo,<br/>the Caputo-Fabrizio and the Atangana-Baleanu fractional-order<br/>derivatives and their applications are also reviewed in the paper.

# Introduction

The subject of fractional calculus (that is, the calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past over four decades, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of mathematical, physical, engineering and statistical sciences. Various operators of fractional-order derivatives as well as fractional-order integrals do indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. Fractional derivatives, fractional integrals and their properties are the subject of study in the field of fractional calculus.

A list of mathematicians, who have provided important contributions up to the middle of 20<sup>th</sup> century, includes Liouville, Riemann, Holmgren, Abel, Fourier, Weyl, Davis, Zygmund, Erdelyi, Kober, Hadamard, Hardy and Littlewood, Grunwald, Letnikov, Riesz. A lot of papers involving the functions of one and more variables have been introduced by many authors Agarwal P. & Purohit S.D. ([1]), Chaurasia and Srivastava ([10]), Chaurasia V. B. L.& Singh Y ([11]), Pandey N. & Khan R. [33], Yashwant Singh & Harmendra Kumar Mandia [42] etc. The well-known books by H.M. Srivastava, K.C. Gupta & S.P. Goyal ([40]), Oldham & Spanier ([25]), Samko, Miller & Ross ([20]), Kilbas & Saigo ([30]) and Nishimoto ([22]) on fractional calculus give a vivid account of its development and use in diverse fields of science and technology.

In the theory of integral equations and problems of mathematical physics, many research works namely Sunil Kumar Sharma & Ashok Singh Shekhawat [47], Hagos Tadesse, D. L. Suthar [46], Goyal, S.P. and Bhagtani, Manita [24], D. L. Suthar, Mitku Andualem, and Belete Debalkie [20], Samko [38], and several others proposed lot of generalizations of fractional integral operators from time to time.

In 1847, Riemann had arrived at expression for fractional integration as-

$$\frac{1}{\Gamma(2)}\int_0^u \frac{f(t)dt}{(u-t)^{1-a}}\,,\qquad x>0$$

This paper was published in 1876.

Most of the theory of fractional calculus is based upon the familiar differential operator defined as-

$${}_{r} D^{w}_{x\{f(x)\}} = \begin{cases} \frac{1}{\Gamma w} \int_{r}^{x} (x-t)^{w-1} f(t) dt & , Re(w) > 0\\ \frac{d^{r}}{dx^{r}} \Big[_{r} D^{w}_{x\{f(x)\}}\Big], & 0 \le Re(w) < r \end{cases} \dots (1.1)$$

Where r is a positive integer.

Case (i): If r = o, the (i) reduces to classical Riemann-Liouville fractional derivatives or integral of order w.

Case (ii): If  $r \to \infty$ , the equation (i) may be defined with the definition of the familiar Wayl fractional operator of order w.

Misra (1981) has defined the fractional derivative operator in the following manner:

$$D_{x}^{\alpha} (\mathbf{x}^{\mu-1}) = \frac{d^{\alpha}}{dx^{\alpha}} \mathbf{x}^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)} \mathbf{x}^{\mu-\alpha-1}, \quad \alpha \neq \mu \qquad \dots (1.2)$$
$$D_{\mathbf{k},\alpha,\mathbf{x}} (\mathbf{x}^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} \mathbf{x}^{\mu+\mathbf{k}}, \quad \alpha \neq \mu+1$$

Erdelyi- Kober fractional integral operators:

$$I^{+} \{f(x)\} = \frac{t^{-\mu-\lambda}}{\Gamma\mu} \int_{0}^{t} x^{\gamma} (t-x)^{\mu-1} f(x) dx \qquad [Re(\mu) > 0, \gamma > -1] \qquad \dots (1.3)$$
$$K^{-} \{f(x)\} = \frac{t^{\delta}}{\Gamma\mu} \int_{t}^{\infty} t^{-\delta-\mu} (x-t)^{\mu-1} f(x) dx \qquad [Re(\mu) > 0, \delta > -1] \qquad \dots (1.4)$$

These operators are generalization of Riemann-Liouville and Weyl fractional integral operator.

# **Different Definitions:**

1. L. Euler (1730) generalized the following formula-

$$\frac{d^n x^m}{dx^n} = m(m-1)\cdots(m-n+1)x^{m-n}$$

by using of the following property of Gamma function,

$$\Gamma(m+1) = m(m-1)\cdots(m-n+1)\Gamma(m-n+1)$$

## **Applications of Fractional Calculus:**

It were renowned mathematicians like Leibniz (1695), Liouville (1834), Riemann (1892) and others who developed recent monographs and symposia proceedings have also exhibited the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetics. Here we some of applications are brought out.

I. First application of fractional calculus was made by Abel in the solution of an integral equation that arises in the formulation of the auto chronous problem. This problem deals with the determination of the shape of a frictionless plane curve through the origin in a vertical plane along which a particle of mass m can fall in a time that is independent of the starting position. If the sliding time is constant T, then the Abel integral equation is

$$\sqrt{2g}T = \int_0^{\eta} (\eta - y)^{-\frac{1}{2}} f'(y) dy,$$

where g is the acceleration due to gravity, and s = f(y) is the equation of the sliding curves. It turns out that this equation is equivalent to the fractional integral equation.

$$T\sqrt{2g} = \Gamma(\frac{1}{2})_0 D_\eta^{-\frac{1}{2}} f'(\eta)$$

#### II. Electric transmission lines

Heaviside successfully developed his operational calculus without rigorous mathematical arguments by introducing the letter p for the differential operator d/dt and gave the solution of the diffusion equation.

$$\frac{\partial^2 u}{\partial x^2} = a^2 p$$

for the temperature distribution u(x,t) in the symbolic form

$$u(x,t) = A \exp(ax\sqrt{p}) + B \exp(-ax\sqrt{p})$$

in which  $p \equiv d/dx$  was treated as constant, where a, A and B are also constant.

#### III. Ultra-sonic wave propagation in human cancellous bone

Fractional calculus is used to describe the viscous interactions between fluid and solid structure.

#### IV. Modeling of speech signals using fractional calculus

The fractional order calculus is using in the modeling of speech signals. The operators can be applied for solve the differential problem.

#### V. Modeling the Cardiac Tissue Electrode Interface Using Fractional Calculus

Conventional lumped element circuit models of electrodes can generalization through modification of the defining current-voltage relationships.

# VI. Application of Fractional Calculus to the sound Waves Propagation in Rigid Porous Materials

The observation that the asymptotic expressions of stiffness and damping in porous materials are proportional to fractional powers of frequency suggests the fact that time derivatives of fractional order might describe the behavior of soundwaves in this kind of materials, including relaxation and frequency dependence.

# I. Using Fractional Calculus for Lateral and Longitudinal Control of Autonomous Vehicles

Fractional Order Controllers (FOC) applied to the path tracking problem in an autonomous electric vehicle. Several control schemes with these controllers have been simulated and compared.

# II. Application of fractional calculus in the theory of visco-elasticity

the fractional derivative method has been used in studies of the complex moduli and impedances for various models of visco-elastic substances.

### III. Fractional differentiation for edge detection

This paper demonstrates how introducing an edge detector based on non-integer (fractional) differentiation can improve the criterion of thin detection, or detection selectivity in the case of parabolic luminance transitions, and the criterion of immunity to noise, which can be interpreted in term of robustness to noise in general.

# IV. Application of Fractional Calculus to Fluid Mechanics

Application of fractional calculus to the solution of time -dependent, viscous-diffusion fluid mechanics problems are presented.

# THE MULTIVARIABLE H-FUNCTION:

The H-function of multivariable introduced by [12], (Srivastava & Panda, 1984). This function is defined and represented in the following manner-

$$H^{0,\lambda:\,(u',v');\,\ldots\ldots,\,;(u^{(r)},v^{(r)})}_{A,C:\,[B',D'];\,\ldots\ldots,\,;[B^{(r)},D^{(r)}]} \cdot \begin{bmatrix} [(a):\theta';\,\ldots\ldots;\theta^{(r)}]:[b':\varphi'];\,\ldots\ldots;[b^{(r)}:\varphi^{(r)}]\\[(c):\psi';\,\ldots\ldots;\psi^{(r)}]:[d':\delta'];\,\ldots\ldots;[d^{(r)}:\delta^{(r)}] \end{bmatrix} Z_1,\ldots,Z_r \end{bmatrix}$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} T(S_1, \dots, S_r) R_1(S_1), \dots, R_r(S_r) Z_1^{S_1} \dots Z_r^{S_1} dS_1 \dots dS_r \qquad \dots (1)$$

Where  $\omega = \sqrt{(-1)}$ 

$$\mathbf{R}_{i}(\mathbf{S}_{i}) = \frac{\prod_{j=1}^{u^{(i)}} \Gamma(\mathbf{d}_{j}^{(i)} - \boldsymbol{\delta}_{j}^{(i)} \mathbf{S}_{i}) \prod_{j=1}^{v^{(i)}} \Gamma(\mathbf{1} - \mathbf{b}_{j}^{(i)} + \boldsymbol{\phi}_{j}^{(i)} \mathbf{S}_{i})}{\prod_{j=u^{(i)}+1}^{D^{(i)}} \Gamma(\mathbf{1} - \mathbf{d}_{j}^{(i)} + \boldsymbol{\delta}_{j}^{(i)} \mathbf{S}_{i}) \prod_{j=v^{(i)}+1}^{B^{(i)}} \Gamma(\mathbf{b}_{j}^{(i)} - \boldsymbol{\phi}_{j}^{(i)} \mathbf{S}_{i})} \qquad \forall (i = 1, 2, 3, ..., r) \quad ...(2)$$

$$\mathbf{T}(\mathbf{S}_{1},\dots,\mathbf{S}_{r}) = \frac{\prod_{j=1}^{\lambda} \mathbf{r}\left(\mathbf{1}-\mathbf{a}_{j}+\sum_{i=1}^{r} \mathbf{\theta}_{j}^{(i)}\mathbf{s}_{i}\right)}{\prod_{j=\lambda+1}^{A} \mathbf{r}\left(\mathbf{a}_{j}-\sum_{i=1}^{r} \mathbf{\theta}_{j}^{(i)}\mathbf{s}_{i}\right)\prod_{j=1}^{C} \mathbf{r}\left(\mathbf{1}-\mathbf{c}_{j}+\sum_{i=1}^{r} \mathbf{\psi}_{j}^{(i)}\mathbf{s}_{i}\right)} \dots (3)$$

and an empty product is interpreted as unity. Suppose, as usual, that the parameters

$$\begin{cases} a_{j}; j = 1, ..., A; b_{j}^{(i)}, j = 1, ..., B^{(i)}; \\ c_{j}; j = 1, ..., C; d_{j}^{(i)}, j = 1, ..., D^{(i)}; \end{cases} \qquad \forall i \in (1, 2, 3, ..., r) \qquad \dots (4)$$

are complex numbers and the associated coefficients

$$\begin{cases} \theta_{j}^{(i)}; j = 1, ..., A; \ \varphi_{j}^{(i)}, j = 1, ..., B^{(i)}; \\ \Psi_{j}^{(i)}; j = 1, ..., C; \ \delta_{j}^{(i)}, j = 1, ..., D^{(i)}; \end{cases} \quad \forall i \in (1, 2, 3, ..., r) \qquad \dots (5)$$

are positive real numbers such that

$$\Omega_{i} = \sum_{j=1}^{A} \theta_{j}^{(i)} - \sum_{j=1}^{C} \Psi_{j}^{(i)} + \sum_{j=1}^{B} \phi_{j}^{(i)} - \sum_{j=1}^{D} \delta_{j}^{(i)} \le 0 \qquad \dots (6)$$

and

+

$$\mathbf{T}_{i} = -\sum_{j=k+1}^{A} \mathbf{\theta}_{j}^{(i)} - \sum_{j=1}^{C} \Psi_{j}^{(i)} + \sum_{j=1}^{v^{(i)}} \mathbf{\phi}_{j}^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}} \mathbf{\phi}_{j}^{(i)}$$
$$+ \sum_{j=1}^{u^{(i)}} \mathbf{\delta}_{j}^{(i)} - \sum_{j=u^{(i)}}^{D^{(i)}} \mathbf{\delta}_{j}^{(i)} > \mathbf{0}, \qquad \forall i \in (1, 2, 3, ..., r) \qquad \dots (7)$$

where the integers  $\lambda$ , A, C, u<sup>(i)</sup>, v<sup>(i)</sup>, B<sup>(i)</sup> and D<sup>(i)</sup> are constrained by the inequalities  $0 \le \lambda \le A, C \ge 0$ ,  $1 \le u^{(i)} \le D^{(i)}$  and  $0 \le v^{(i)} \le B^{(i)} \forall i \in (1,2,3,...,r)$ , and the inequalities in (1.9) hold true for suitably restricted value of the complex variables  $Z_1,...,Z_r$ 

The sequence of parameters in (1.4) is such that none of poles of the integrand coincide, that is, the poles of the integrand in (1.4) are simple.

The contour L<sub>i</sub> in the complex S<sub>i</sub>-plane is of the Mellin–Barns type which runs from  $-\omega\infty$  to  $+\omega\infty$ , with identifications, if necessary, to ensure that all the poles of  $\Gamma\left(d_j^{(i)} - \delta_j^{(i)}S_i\right)$ ,  $j=1,...,u^{(i)}$  are separated from those of  $\Gamma\left(1 - b_j^{(i)} + \phi_j^{(i)}S_i\right)$ ,  $j=1,...,v^{(i)}$  and  $\Gamma\left(1 - a_j + \sum_{i=1}^r \theta_j^{(i)}S_i\right)$ ,  $j=1,...,\lambda$ 

$$\forall i \in \{1, 2, 3, \dots, r\}.$$

Then it is known that the multiple Mellin–Barnes contour integral in (1.4) converges absolutely under the condition (1.10), when

$$|\arg(\mathbf{Z}_{i})| < \frac{1}{2} T_{i} \pi, \quad \forall i \in \{1, 2, 3, ..., r\}$$
 ...(8)

The points  $Z_i=0$  (i=1,...,r) and various exceptional parameter values being tacitly excluded.

For the multivariable H- function given in (1.4), we shall employ the following contracted notation:

$$H [Z_1,...,Z_r] \text{ or } H^{0,\lambda:(u',v'); \ldots, \ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D']; \ldots, \ldots;[B^{(r)},D^{(r)}]} \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix} \text{ or } H \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix}$$
(9)

Furthermore, we have the known asymptotic expansions in the following form:

$$H[Z_{1},...,Z_{r}] = \begin{cases} O(|Z_{1}|^{\Delta_{1}} ... |Z_{r}|^{\Delta_{r}}), Max(|Z_{1}| ... |Z_{r}|) \to 0\\ O(|Z_{1}|^{\nabla_{1}} ... |Z_{1}|^{\nabla_{r}}), \lambda = 0, Min(|Z_{1}| ... |Z_{r}|) \to \infty \end{cases}$$
...(10)

Where,

$$\Delta_{i} = \operatorname{Min}\left\{ \operatorname{R}_{e}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right) \right\}, \left(j = 1, 2, ..., u^{(i)}\right) \qquad \dots (11)$$

$$\nabla_{i} = Max \left\{ R_{e} \left( \frac{b_{j}^{(i)} - 1}{\phi_{j}^{(i)}} \right) \right\}, (j = 1, 2, ..., v^{(i)}), \forall i \in \{1, 2, 3, ..., r\}$$
...(12)

Again, throughout the present work, we employ in abbreviation (a) to denote the sequence of A parameters  $a_1, \ldots, a_A$ ; for each  $i=1,\ldots,r$ ,  $(b^{(i)})$  abbreviates the sequence of  $B^{(i)}$  parameters,  $b_j^{(i)}$ ,  $j=1,\ldots,B^{(i)}$  with similar interpretations for (c),  $(d^{(i)})$ , etc.  $i=1,\ldots,r$ ; it will be understood for example, that  $b^{(1)}=b'$ ,  $b^{(2)}=b''$  and so on.

Also, for the sake of brevity, we use the following contracted notations:

$$[(\mathbf{a})]_{\mathbf{n}} = \prod_{j=1}^{A} [\mathbf{a}_{j}]_{\mathbf{n}}, [(\mathbf{b}^{(i)})]_{\mathbf{n}} = \prod_{j=1}^{B^{(i)}} [\mathbf{b}_{j}^{(i)}]_{\mathbf{n}}, i=1,2,3,...r \text{ etc.} ...(13)$$

#### Conclusion

In this paper, we reviewed various definitions of fractional derivatives and fractional integrals and applications are also reviewed. The application of fractional calculus was made by Abel in the solution of an integral equation. This problem deals with the determination of the shape of a frictionless plane curve through the origin in a vertical plane along which a particle of mass m can fall in a time that is independent of the starting position. Fractional calculus is used to describe the

viscous interactions between fluid and solid structure. Comparing the fractional results for boundary shear -stress and fluid speed to the existing analytical results for the first and second Stokes problems.

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# References

- 1. Agarwal, P., & Purohit, S. D. (2013). The unified pathway fractional integral formulae. *Journal of fractional calculus and applications*, 8.
- 2. Chaurasia, V. B., & Shrivastava, A. (2007). An Unified Approach to Fractional Calculus Pertaining to H- functions. *Soochow Journal of Mahematics*.
- 3. Chaurasia, V. B., & Singh, Y. (2012). New Triple Integral Relation associated with H-function multivariable H-function. *International Journal of Mathematical Archieve*, 9.
- 4. Singh, Y., & Mandiya , H. K. (2013). Some properties and series representation of Hfunction of two variables.
- 5. Shrivastava, H. M., Gupta, K. C., & Goyal, S. (1982). The H- function of one and two variables with applications. *South Asian Publishers*.
- 6. Oldhalm, K., & Spanier, J. (1970). The Replacement of Fick's Lawby a formulation involving semi differentiation. *J. Electronal*.
- 7. Miller, K. S., & Ross, B. (1993). An Introduction to the fractional calculus and fractional differential equations.
- 8. Samko, S. G., k, & Kilbas, A. A. (1993). Fractional Integrals and Derivatives: Theory and Applications.
- 9. Nishimoto, K. (1990). Proceedings of the international conference on fractional calculus and it's applications.
- 10. Tadesse, H., Suthar, D. L., & Ayalew, M. (2019). Finite Integral Formulas Involving Multivariable Aleph-Functions. *Hindawi Journal of Applied Science*, 10.
- 11. Suthar, D. L., Andualem, M., & Debalkie, B. (2019). A Study on Generalized Multivarible Mittag- Leffer Function via Generalized Fractional Calculus operators. *Hindawi Journal of Mathematics*, 7.
- 12. Shrivastava, H. M., & Panda, R. (1984). An Application of the fractional derivatives.
- 13. Magin, R. L. (2004). Fractional Calculus in Bio Engineering, Part1, Crit Rev BiomedEng, 104.
- Magin, R. L. (2008). Modeling the Cardiac Tissue Electrode Interface using Fractional Calculus, Journal of Vibration and Control, Vol. 14, 10