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EXISTENCE RESULT FOR THE ANTI PERIODIC BOUNDARY VALUE PROBLEMS OF FRACTIONAL ORDER $0 < \alpha < 3$

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Abstract. This paper studies existence and uniqueness of solutions for system of fractional differential equations involving Caputo derivative with anti periodic boundary conditions of order $\alpha \in (0.3)$. We obtain the result by using Banach fixed point theorem.
Keywords. *Caputo fractional derivative, fractional differential equations, anti-periodic boundary conditions, Banach fixed point theorem.*

1 Introduction

In recent years the subject of fractional calculus gained much momentum and attracted many researchers and mathematicians. Considerable interest in field of fractional calculus has been developed by the applications to different areas of applied science and engineering like physics, biophysics, aerodynamics, control theory, visco-elasticity, capacitor theory, electrical circuit, description of memory and hereditary properties etc.

Anti periodic boundary value problems constitute an important class of boundary value problems and have recently received considerable attention. Anti periodic boundary conditions occur in mathematical modeling of many physical processes, see [6] – [10] and references therein.

The Banach fixed point theorems is used [11] to investigate existence and uniqueness of for integro differential equations of fractional order $\alpha \in (1,2)$ with antiperiodic boundary conditions. In [7] the author investigated existence problem of anti periodic boundary value problem to fractional differential equation for $\alpha \in (2,3)$ by using Banach fixed point. Motivated by these works we study in this paper the existence of solution to fractional differential equation when $\alpha \in (0,3]$ with anti periodic boundary conditions. Precisely we consider the following problem

$$\begin{cases} {}^{c}D^{\alpha_n} x_n = f_n(t, x_n(t)),\ t \in [0,T],\ T > 0 \\ 0 < \alpha_n \le 3, n = 1,2,3 \\ x(0) = -x(T), x'(0) = -x(T),\ x^n(0) = -x^n(T) \end{cases} \tag{1}$$

where ${}^{c}D^{\alpha_n}$ denotes the Caputo fractional derivative of order α_n and f is a continuous function

2. Preliminaries

Definition 2.1. A real function f(t) is said to be in the space C_p, $\mu \in \Re$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0,\infty]$, and it is said to be in the space C_p^n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

Definition 2.2. A function $f \in C_\mu$, $\mu \ge -1$ is said to be functional integrable of order $\alpha > 0$ if

$$(I^\alpha f)(t) = I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

and if $\alpha_n = 0$ then $I^0 f(t) = f(t)$.

Next we introduce the Caputo fractional derivative.

Definition 2.3. Caputo fractional derivative is defined as

$$(D^\alpha f)(t) = D^\alpha f(t) = I^{n-\alpha} \frac{d^n f}{dt^n}(t) = \frac{1}{\Gamma n - \alpha} \int (t-s)^{n-1} f(s)ds$$

for $n-1 < \alpha < n, n \in N, t > 0, f \in C^n$

Lemma 2.4. [3]. For $\alpha > 0$ the solution of fractional differential equation ${}^{c}D^\alpha x(t) = 0$ is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1} \tag{2}$$

where $c_i \in \Re, i = 1,2,\ldots,n-1 (n = [\alpha]+1)$ where $[\alpha]$ denotes the integer part of $\alpha > 0$.

To study the nonlinear problem (1) we need following lemma.

Lemma 2..5. For any $\varphi \in C[0,T]$ the unique solution of bounded value problem.

$$\begin{cases} {}^{c}D^{\alpha_n} x_n(t) = f_n(t, x_n(t)),\ t \in [0,T],\ T\ .\ 0 \\ \quad 0 < \alpha_n \leq 3, n = 1,2,3 \\ x(0) = -x(T), x'(0) = -x'(T), x^n(0) = -x^n(T). \end{cases} \tag{3}$$

is

$$x_n(t) = \int_0^T G_n(t,s)\varphi(s)\,ds$$

where $G_n(t,s)$ is Green's function corresponding to α_n.

$$G_1(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha_1 - 1} - \frac{1}{2}(T-s)^{\alpha_1 - 1}}{\Gamma(\alpha)}, & 0 < s < t < T \\ -\dfrac{(T-s)^{\alpha_1 - 1}}{2\Gamma\alpha_1} & 0 < t < s < T \end{cases} \tag{4}$$

$$G_2(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha_2 - 1} - \frac{1}{2}(T-s)^{\alpha_2 - 1}}{\Gamma\alpha_2} + \dfrac{(T-2t)(T-s)^{\alpha_2 - 2}}{4\Gamma\alpha_2 - 1}, & 0 < s < t < T \\ \dfrac{(T-s)^{\alpha_2 - 1}}{2\Gamma\alpha_2} + \dfrac{(T-2t)(T-s)^{\alpha_2 - 2}}{4\Gamma\alpha_2 - 1}, & 0 < t < s < T \end{cases} \tag{5}$$

$$G_3(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha_3 - 1} - \frac{1}{2}(T-s)^{\alpha_3 - 1}}{\Gamma\alpha_3} + \dfrac{(T-2t)(T-s)^{\alpha_3 - 2}}{4\Gamma\alpha_3 - 1} + \dfrac{t(T-t)(T-s)^{\alpha_3 - 3}}{4\Gamma\alpha_3 - 2}, & 0 < s < t < T \\ \dfrac{(T-s)^{\alpha_3 - 1}}{2\Gamma\alpha_3} + \dfrac{(T-2t)(T-s)^{\alpha_3 - 3}}{4\Gamma\alpha_3 - 2}, & 0 < t < s < T \end{cases} \tag{6}$$

Proof. By using Lemma 2.4 for some constants c0, c_1, c_2 we have for $0 < \alpha \leq 1$

$$x_1(t) = \int_0^t \frac{(t-s)^{\alpha_1 - 1}}{\Gamma\alpha_1} \varphi_1(s)\,ds - c_0$$

at $t = 0$ we have $x_1(0)$ at $t = T$

$$xT = \int_0^T \frac{(T-s)^{\alpha_1 - 1}}{\Gamma\alpha_1} \varphi_1(s)ds = c_0$$

by using boundary condition $x(0) = -x(T)$ we have

$$c_0 = \int_0^T \frac{(T-s)^{\alpha_1 - 1}}{\Gamma\alpha_1} \varphi_1(s)\,ds$$

hence

$$x_1(t) \backslash \int_0^t \frac{(t-s)^{\alpha_1 - 1}}{\Gamma\alpha_1} - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha_1 - 1}}{\Gamma\alpha_1} \varphi_1(s)\,ds$$

$$x_1(t)=\int_0^T G_1(t,s)\,\varphi_1(s)\,ds.$$

The Green's function is:

$$G_1(t,s)=\begin{cases}\dfrac{(t-s)^{\alpha_1-1}-\frac{1}{2}(T-s)^{\alpha_1-1}}{\Gamma\alpha_1}, & 0<s<t<T\\ -\dfrac{(T-s)^{\alpha_1-1}}{2\Gamma\alpha_1} & 0<t<s<T\end{cases} \tag{7}$$

Similarly for $1<\alpha\le 2$

$$x_2(t)=\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma\alpha_2}\varphi_2(s)ds-c_0-c_1t$$

and

$$x'_2(t)=\int_0^t \frac{(t-s)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\varphi_2(s)\,ds-c_1.$$

By using boundary conditions

$$x_2(t)=\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2-1)}\varphi_2(s)ds-\frac{1}{2}\int_0^T \frac{(T-s)^{\alpha_2-1}}{\Gamma\alpha_2}\varphi_2(s)ds+\int_0^T \frac{(T-2t)(T-s)^{\alpha_2-2}}{4\Gamma(\alpha_2-1)}\varphi_2(s)ds$$

$$=\int_0^T G_2(s,t)\varphi_2(s)\,ds. \tag{8}$$

$$G_2(t,s)=\begin{cases}\dfrac{(t-s)^{\alpha_2-1}-\frac{1}{2}(T-s)^{\alpha_2-1}}{\Gamma\alpha_2}+\dfrac{(T-2t)(T-s)^{\alpha_2-2}}{4\Gamma\alpha_2-1}, & 0<s<t<T\\ \dfrac{(T-s)^{\alpha_2-1}}{2\Gamma\alpha_2}+\dfrac{(T-2t)(T-s)^{\alpha_2-2}}{4\Gamma\alpha_2-1}, & 0<t<s<T\end{cases} \tag{9}$$

Finally for $2<\alpha_3\le 3$

$$x_3(t)=\int_0^T G_3(t,s)\varphi_3(s)\,ds$$

where

$$G_3(t,s)=\begin{cases}\dfrac{(t-s)^{\alpha_3-1}-\frac{1}{2}(T-s)^{\alpha_3-1}}{\Gamma\alpha_3}+\dfrac{(T-2t)(T-s)^{\alpha_3-2}}{4\Gamma\alpha_3-1}+\dfrac{t(T-t)(T-s)^{\alpha_3-3}}{4\Gamma\alpha_3-2}, & 0<s<t<T\\ \dfrac{(T-s)^{\alpha_3-1}}{2\Gamma\alpha_3}+\dfrac{(T-2t)(T-s)^{\alpha_3-3}}{4\Gamma\alpha_3-2}, & 0<t<s<T\end{cases} \tag{10}$$

3. Existence result

The existence problem to the given fractional nonlinear differential system with anti-periodic boundary conditions is investigated in this section by using well known Banach fixed point theorem.

Let C = C[0,1],$\Re$ denote the Banach space of all continuous functions from $[0,1] \to R$ endowed with the norm defined by $||x|| = \sup |x)t,t \in [0,1]) |$.

Now we state some known results to prove the existence of solution of (1).

Theorem 3.1. Let X be a Banach space and Ω is an open and bounded subset of X and let T : $\Omega \to X$ and $||Tu|| \le ||u||$, for all $u \in \Omega$. Then T has a fixed point in Ω.

Theorem 3.2. Define an operator

$g_n : C \to C$ as n = 1,2,3 and $t \in [0,1]$ for $n = 1, 0 < \alpha_1 \le 1$.

$$(g_1x)(t) = \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma\alpha_1}\phi_1(s,x(s))ds - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha_1-1}}{\Gamma\alpha_1}\phi_1(s,x(s))ds \tag{11}$$

for $n = 2, 1 < \alpha_2 \le 2$.

$$\begin{aligned}(g_2x)(t) = &\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma\alpha_2}\phi_2(s,x(s))ds - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha_2-1}}{\Gamma\alpha_2}\phi_2(s,x(s))ds \\ &+ \frac{1}{2}\int_0^T \frac{(T-2t)(T-s)^{\alpha_2-2}}{4\Gamma(\alpha_2-1)}\phi_2(s,x(s))ds\end{aligned} \tag{12}$$

for $n = 3, 2 < \alpha_3 \le 3$

$$\begin{aligned}(g_3x)(t) = &\int_0^t \frac{(t-s)^{\alpha_3-1}}{\Gamma\alpha_3}\phi_3(s,x(s))ds - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha_3-1}}{\Gamma\alpha_3}\phi_3(s,x(s))ds \\ &+ \frac{(T-2t)}{4}\int_0^T \frac{(T-s)^{\alpha_3-2}}{\Gamma(\alpha_3-1)}\phi_3(s,x(s))ds \\ &+ \frac{t(T-t)}{4}\int_0^T \frac{(T-s)^{\alpha_3-3}}{\Gamma(\alpha_3-2)}\phi_3(s,x(s)ds,\ t\in[0,1]\end{aligned} \tag{13}$$

Observe that problem (1) has a solution if and only if the operator g_n has a fixed point.

Lemma 3.3. The operator $g_n : C \to C$ is completely continuous.

Proof. Let $\Omega \subset C$ be bounded then $\forall t \in [0,1]$, $x \in \Omega$, there exists a positive contant L_n such that

$$|\phi_n(t,x)| \le L_n, n = 1,2,3.$$

Thus for $0 < \alpha_1 \le 1$, we have

$$(g_1x)(t) = \int_0^t \frac{(t-s)^{\alpha_1 - 1}}{\Gamma\alpha_1} | \phi_1(s, x(s))ds + \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha_1 - 1}}{\Gamma\alpha_1} \phi_1(s, x(s)) | ds$$

$$\leq \left[\frac{1}{\Gamma\alpha_1} \int_0^t (t-s)^{\alpha_1 - 1} ds + \frac{1}{2\Gamma\alpha_1} \int_0^T (T-s)^{\alpha_1 - 1} ds \right]$$

$$\leq L_1 \left[\frac{T_{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \frac{T^{\alpha_1}}{2\Gamma(\alpha_1)} \right]$$

$$= \frac{3T^{\alpha_1}}{2\Gamma(\alpha_1 + 1)} . L_1$$

$$= M_1 L_1$$

where $M_1 = \frac{3T^{\alpha_1}}{2\Gamma(\alpha_1 + 1)}$ which implies that

$$\| g(x) \| \leq M_1 L_1. \quad (14)$$

Furthermore

$$(g_1x)'(t) = \int_0^t \frac{(t-s)^{\alpha_1 - 2}}{\Gamma\alpha_1 - 1} | \phi_2(s, x(s))ds + \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha_1 - 2}}{\Gamma\alpha_1} \phi_2(s, x(s)) | ds$$

$$\leq L_1 \left[\int_0^t \frac{(t-s)^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} ds + \frac{1}{2}\int_0^T \frac{(eT-s)^{\alpha_1 - 1}}{\Gamma(\alpha_1 - 1)} ds \right]$$

$$\leq L_1 \left[\frac{3T^{\alpha_1 - 1}}{2\Gamma\alpha_1} \right]$$

$$= M_1' L_1$$

where $M_1' = \frac{3T^{\alpha_1 - 1}}{2\Gamma\alpha_1}$ which implies that

$$\| (g_1x)'(t) \| \leq M_1' L_1. \quad (15)$$

Hence for t_1, $t_2 \in [0,t]$, we have

$$| (g_1x)(t_1) - (g_1x)(t_2) | \leq \int_{t_1}^{t_2} \{(g_1x)'(s) | ds \leq M_1' L_1 (t_1 - t_2). \quad (16)$$

This implies that g_1 is equicontinuous on [0,1]. By Arzela Ascoli theorem we can say that $g_1 : C \to C$ is completely continuous.

In the similar manner we can prove that $g_2 : C \to C$ and $g_3 : C \to C$ are completely continuous for $1 < \alpha_2 \leq 2$ and $2 < \alpha_3 \leq 3$ respectively.

Hence we can say $g_n : C \to C$ is completely continuous for $0 < \alpha_n \le 3$ where n = 1,2,3.

Now we prove existence and uniqueness result by means of Banach fixed point theorem.

Theorem 3.4. Assume $\phi : [0,1] \times X \to X$ is a continuous function satisfying the condition

$$\|\phi_n(t,x) - \phi_n(t,y)\| \le L_n \|x - y\|, \forall t \in [0,T], x, y \in X, \ n = 1,2,3$$

with

$$L_1 \le \frac{2\Gamma(\alpha_1 + 1)}{3T^{\alpha_1}}$$

$$L_2 \le \frac{2\Gamma(\alpha_2 + 1)}{T^{\alpha_2}}\left(3 + \frac{\alpha_2}{2}\right)$$

$$L_3 \le \frac{2\Gamma(\alpha_3 + 1)}{T^{\alpha_3}}\left(3 + \frac{\alpha_3^2}{2}\right).$$

Proof. Setting sup $t \in [0,1]$ $|\phi_i(t,0)| = M_i$ and selecting $r_i \ge \frac{M_i}{B_i}$ where i = 1,2,3 and L_1, L_2, L_3 as defined above. We show that $gB_{r_i} \subset B_{r_i}$ where

$$B_{r_i} = \{x \in C : \|x\| \le r_i\}$$

for $x_i \in B_{r_i}$, we have

$$\|g_1 x)(t)\| \le \max_{t \in [0,T]}\left[\int_0^t \frac{(t-s)^{(\alpha_1 - 1)}}{\Gamma\alpha_1} |\phi_1(s, x(s))|\, ds + \frac{1}{2}\int_0^T \frac{(T-s)^{(\alpha_1 - 1)}}{\Gamma\alpha_1} |\phi_1(s, x(s)) ds|\right]$$

$$\le \max_{t \in [0,T]}\left[\int_0^t \frac{(t-s)^{(\alpha_1 - 1)}}{\Gamma\alpha_1} (|\phi_1(s, x(s)) - \phi_1(s,0) + \phi_1(s,0)|) ds\right]$$

$$+ \frac{1}{2}\int_0^T \frac{(T-s)^{(\alpha_1 - 1)}}{\Gamma\alpha_1} |\phi_1(s, x(s)) - \phi_1(s,0) + \phi_1(s,0)|) ds$$

$$\le (L_1 r_1 + M_1) \max_{t \in [0,T]}\left[\frac{1}{\Gamma\alpha_1}\int_0^t (t-s)^{\alpha_1 - 1} ds + \frac{1}{2\alpha_1}\int_0^T (T-s)^{\alpha_1 - 1} ds\right]$$

$$\le (L_1 r_1 + M_1)\frac{3T^{\alpha_1}}{2\Gamma(\alpha_1 + 1)}$$

$$\le r_1$$

which implies that

$$\|g_1(x)(t)\| \le r_1. \qquad (17)$$

By using the same argument we can prove that

$$\|(g_i x)(t)\| \le r_i, r = 1,2,3 \tag{18}$$

Now for x, y $\in$ C and for each t $\in$ [0,1] for $0 < \alpha_1 \le 1$ we obtain

$$\| g_1 x)(t) - (g_1 y)(t)| \le \max_{t\in[0,T]} \left[\int_0^t \frac{(t-s)^{(\alpha_1 - 1)}}{\Gamma \alpha_1} \| \phi_1(s, x(s))| \, ds - \phi_1(s, y(s))\| \right.$$

$$\left. + \frac{1}{2} \int_0^T \frac{(T-s)^{(\alpha_1 - 1)}}{\Gamma \alpha_1} \| \phi_1(s, x(s)) - \phi_1(s, y(s))\| ds \right]$$

$$\le L_1 \| x - y \| \max_{t\in[0,t]} \left[\frac{1}{\alpha_1} \int_0^t (t-s)^{\alpha_1 - 1} ds + \frac{1}{2\alpha_1} \int_0^T (T-s)^{\alpha_1 - 1} ds \right]$$

$$\le \frac{3L_1 T^{\alpha_1}}{2\Gamma(\alpha_1 + 1)} \| x - y \|$$

$$= \Lambda_{L_1, T, \alpha_1} \| x - y \|$$

which implies that

$$\|(g_1 x)(t) - (g_1 y)(t)\| \le \Lambda_{L_1, T, \alpha_1} \| x - y \| \tag{19}$$

where $\Lambda_{L_1, T, \alpha_1} = \frac{3L_1 T^{\alpha_1}}{2\Gamma(\alpha_1 + 1)}$ which depends upon only on parameters involved in the problem.

As $\Lambda_{L_1, T, \alpha_1} < 1$ hence g_1 is a contraction.

Now for $1 \le \alpha_1 \le 2$

$$\| g_2 x)(t) - (g_2 y)(t)\| \Lambda_{L_2, T, \alpha_2} \tag{20}$$

where $\Lambda_{L_2, T, \alpha_2} = \frac{T^{\alpha_2\left(3 + \frac{\alpha_2}{2}\right)}}{2\Gamma(\alpha_2 + 1)}$ As $\Lambda_{L_2, T, \alpha_2} < 1$ hence g_2 is contraction.

Finally for $2 < \alpha_3 \le 3$

$$\|(g_3 x)(t) - (g_3 y)(t)\| \le \Lambda_{L_3, T, \alpha_3} \tag{21}$$

where $\Lambda_{L_3, T, \alpha_3} = \frac{T^{\alpha_3\left(3 + \frac{\alpha_3^2}{2}\right)}}{2\Gamma(\alpha_3 + 1)}$. Again $\Lambda_{L_3, T, \alpha_3} < 1$ hence g_3 is also contraction.

Thus conclusion of the theorem follows by Contraction mapping principle or Banach fixed point theorem.

4. Example

Example 4.1. Now consider a fractional system of equation with anti periodic boundary conditions

$$\begin{pmatrix} D^{\frac{1}{2}}x_1(t)=\dfrac{x_1(t)}{(2+t)^3}, & x(0)=-x(2) \\ D^{\frac{3}{2}}x_2(t)=\dfrac{x_2(t)}{(7\_e^t)(1+x_2(t))}, & x(0)=-x,(2),x'(0)=-x'(2) \\ D^{\frac{3}{2}}x_3(t)=\dfrac{x_3(t)}{(2+t^3)(1+x_3(t))}, & x(0)=-x(2),x'(0)=-x'(2),x''(0)=x''(2) \end{pmatrix}$$

where t ∈ [0,2].

Solution 4.2. Here T = 2 in each case $L_i=\frac{1}{8}$ for i = 1,2,3 in each case we have

$$\|\phi_n(t,s))-\phi_n(t,y)\|\le\frac{1}{8}\|x-y\|, n=1,2,3$$

By using Theorem 3.4 for $\alpha_1=\frac{1}{2}, 0<\alpha_1\le 1$

$$\Lambda_{L_1,T,\frac{1}{2}}=\frac{2L_1T^{\frac{1}{2}}}{3\Gamma\left(1+\frac{1}{2}\right)}=0.1880631945<1.$$

For $\alpha_2=\frac{3}{2}, 1<\alpha_2\le 2$

$$\Lambda_{L_2,T,\frac{3}{2}}=\frac{T^{\frac{3}{2}}L_2}{2\Gamma\frac{5}{2}}\left(3+\frac{3}{4}\right)=0.1003003704<1.$$

For $\alpha_3=\frac{5}{2}, 2<\alpha_3\le 3$

$$\Lambda_{L_3,T,\frac{5}{2}}=\frac{T^{\frac{5}{2}}L_3}{2\Gamma\frac{7}{2}\left(3+\frac{25}{8}\right)}=0.02456335602<1.$$

In all cases $\Lambda_{L_i,T,\alpha_n}<1.$

Hence by Banach fixed point theorem and Theorem 3.4 the system of differential equations of fractional order $0 < \alpha_n \leq 3$ with anti periodic boundary conditions has a unique solution.

5 Conclusion

As existence result is given for system of fractional differential equation involving Caputo derivative with anti periodic boundary conditions of order $\alpha \in (0.3)$ by using Banach fixed point theorem

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