



Review Article

An overview of Fractional Calculus and Its Applications

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Abstract

Keywords

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The concept of operators that involve integration and differentiation of arbitrary order provides an effective approach for analyzing complex phenomena and dynamic processes. This review paper presents a detailed study of several fractional operators and quantum operators developed by different researchers. We see that arbitrary-order models outperform classical ordinary-order models, especially for systems with memory effects and time-varying behaviour. Therefore, fractional calculus, q-calculus, and quantum calculus have become important and continue to grow rapidly due to their extensive use across mathematics, physics, engineering, and the applied sciences. The paper uses integral transforms and analytical techniques to obtain useful results. Several special functions are studied to extend the classical results and establish connections among various mathematical methods. These functions are also useful for studying the applications of fractional operators in quantum mechanics, reaction-diffusion systems and complex physical models.

1 Introduction to Fractional Calculus

Fractional Calculus (F C) is a generalization of ordinary calculus: it extends the concepts of differentiation and integration to arbitrary orders. Modern mathematics F C provides a more effective way to investigate complex systems than conventional mathematical approaches. It is useful for modelling systems with memory and provides more flexible, generalized solutions than integer-order calculus.

It is widely used to solve real-world problems and to improve understanding of dynamic systems and irregular processes.

1.1 Scope and Contribution of the Review

Fractional calculus has experienced remarkable growth over the past decades, leading to the development of numerous fractional operators, generalized integral transforms, and applications in science and engineering. Although several review articles have

focused on specific classes of fractional operators or particular application domains, a comprehensive review that critically compares different operator families together with their mathematical properties, computational aspects, and practical application remains limited.

The present review aims to provide a comprehensive overview of the fundamental and generalized fractional operators, including the Riemann-Liouville, Caputo, Hilfer, Erdélyi-Kober, Weyl, Saigo, and Marichev-Saigo-Maeda operators, along with recent developments in generalized, variable-order, proportional, and non-singular kernel fractional operators. In addition, the review discusses their relationships with important special functions and highlights representative applications in quantum mechanics, reaction-diffusion systems, biological modelling. Unlike many existing survey articles that primarily summarize definitions or focus on a single operator family, this review emphasizes a comparative analysis of the advantages, limitations, mathematical characteristics, and application domains of various fractional operators. Furthermore, recent advances, computational techniques, research gaps, and future research directions are discussed to provide researchers with a broader perspective on the current state and future potential of fractional calculus.

2 Core Foundations of Fractional Calculus

Within this historical timeline, S. F. Lacroix (Lacroix et al. 1819) holds a special place, as he was the first to provide a specific result for a fractional derivative. He used the Gamma function to extend the basic rule of differentiation to a frac-

tional order, showing how to find the derivative of x^r for any order defined as:

$$\frac{d^s}{dx^s} x^r = \frac{r!}{(r-s)!} x^{r-s}, \quad r \in \mathbb{N}; \quad s \in \mathbb{N}_0; \quad r \geq s. \quad (1)$$

Riemann-Liouville Operator ($\mathcal{RLO}$) (Canavati et al. 1987) is defined as:

$$\begin{aligned} {}_b I_s^\varrho f(s) &= {}_b D_s^{-\varrho} f(s) \\ &= \frac{1}{\Gamma(\varrho)} \int_b^s (s-y)^{\varrho-1} f(y) dy, \end{aligned} \quad (2)$$

where ϱ indicates a freely chosen fixed point.

Erdélyi-Kober Operators ($\mathcal{EKO}$) (Herrmann et al. 2014) are defined as the following:

$$\begin{aligned} E_{0,s}^{\varrho,\nu} f(s) &= I_{0,s}^{\varrho,\nu} f(s) \\ &= \frac{s^{-\varrho-\nu}}{\Gamma(\varrho)} \int_0^s (s-y)^{\varrho-1} y^\nu f(y) dy, \end{aligned} \quad \mathbb{R}(\varrho) > 0, \quad (3)$$

and

$$\begin{aligned} K_{s,\infty}^{\varrho,\nu} f(s) &= J_{s,\infty}^{\varrho,\nu} f(s) \\ &= \frac{s^\nu}{\Gamma(\varrho)} \int_s^\infty (y-s)^{\varrho-1} y^{-\varrho-\nu} f(y) dy, \end{aligned} \quad \mathbb{R}(\varrho) > 0. \quad (4)$$

Weyl fractional integral operators ($\mathcal{WFO}$) (Samko et al. 1993) are defined as:

$$\begin{aligned} {}_s \mathcal{W}_\infty^\varrho f(s) &= \frac{1}{\Gamma(\varrho)} \int_s^\infty (y-s)^{\varrho-1} f(y) dy, \\ &\quad -\infty < s < \infty, \end{aligned} \quad (5)$$

and

$${}_{-\infty}\mathcal{W}_s^\varrho f(x) = \frac{1}{\Gamma(\varrho)} \int_{-\infty}^s (s-y)^{\varrho-1} f(y) dy, \quad -\infty < s < \infty. \quad (6)$$

The Caputo fractional derivative ($\mathcal{CFD}$) (Caputo et al. 1967) is defined as follows:

$${}^C D_t^\varrho f(t) = \frac{1}{\Gamma(m-\varrho)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\varrho-m+1}} d\tau, \quad m-1 < \varrho < m. \quad (7)$$

Hilfer fractional derivative ($\mathcal{HFD}$) (Hilfer et al. 2000) is defined as follows:

$$D_{c+}^{S,\varrho} f(u) = I_{c+}^{S(1-\varrho)} \frac{d}{du} \left(I_{c+}^{(1-S)(1-\varrho)} f(u) \right), \quad (8)$$

where $0 < S < 1$, $0 \leq \varrho \leq 1$ and control the transition between $\mathcal{RLO}$ ($\varrho = 0$) and $\mathcal{CFD}$ ($\varrho = 1$).

The Saigo Operator ($\mathcal{SO}$) (Saigo et al. 1978) is defined as:

$$\begin{aligned} \left(I_{0+}^{\varrho, \nu, \zeta} f \right) (s) &= \frac{s^{-\varrho-\nu}}{\Gamma(\varrho)} \int_0^s (s-r)^{\varrho-1} \\ &\times {}_2F_1\left(\varrho + \nu, -\zeta; \nu; 1 - \frac{r}{s}\right) \\ &\times f(r) dr, \quad (\Re(\varrho) > 0), \end{aligned} \quad (9)$$

and

$$\begin{aligned} \left(I_{-}^{\varrho, \nu, \zeta} f \right) (s) &= \frac{1}{\Gamma(\varrho)} \int_s^\infty (r-s)^{\varrho-1} r^{-\varrho-\nu} \\ &\times {}_2F_1\left(\varrho + \nu, -\zeta; \varrho; 1 - \frac{s}{r}\right) \\ &\times f(r) dr, \quad (\Re(\varrho) > 0). \end{aligned} \quad (10)$$

The Marichev-Saigo-Maeda Operator ($\mathcal{MSMO}$)

(Samko et al. 1993) is defined as:

$$\begin{aligned} \left(I_{0,s}^{\varrho, \varrho', \nu, \nu', \zeta} f \right) (s) &= \frac{s^{-\varrho}}{\Gamma(\zeta)} \int_0^s (s-r)^{\zeta-1} r^{-\varrho'} \\ &\times F_3\left(\varrho, \varrho', \nu, \nu', \zeta; 1 - \frac{r}{s}, 1 - \frac{s}{r}\right) \\ &\times f(r) dr. \end{aligned} \quad (11)$$

and

$$\begin{aligned} \left(I_{s,\infty}^{\varrho, \varrho', \nu, \nu', \zeta} f \right) (s) &= \frac{s^{-\alpha'}}{\Gamma(\zeta)} \int_s^\infty (r-s)^{\zeta-1} r^{-\zeta'} \\ &\times F_3\left(\varrho, \varrho', \nu, \nu', \zeta; 1 - \frac{s}{r}, 1 - \frac{r}{s}\right) \\ &\times f(r) dr. \end{aligned} \quad (12)$$

under the state of conditions $\varrho, \varrho', \nu, \nu', \zeta \in \mathbb{C}$ ($\Re(\zeta) > 0$) and $s > 0$.

2.1 Comparison of Fractional Operators

Classical operators, such as the Riemann-Liouville and Caputo operators, provide a rigorous mathematical foundation while generalized operators, including the Saigo and Marichev–Saigo–Maeda operators, offer greater flexibility by encompassing several existing operators as special cases, although their mathematical formulations and numerical implementations are comparatively more complex. Therefore, the choice of a fractional operator should be guided by the nature of the problem, the required initial conditions, and the desired modelling accuracy.

Table-1 compares the major fractional operators discussed in this review with respect to their advantages, limitations, and representative applications.

Table 1. Comparison of Major Fractional Operators

Operator	Advantages	Limitations	Applications
Riemann-Liouville	Strong mathematical foundation	Fractional initial conditions	Diffusion, Viscoelasticity
Caputo	Physical initial conditions	Singular kernel	Engineering, Control
Hilfer	Interpolates RL and Caputo	Additional parameter	Control, Heat Transfer
Erdélyi-Kober	Generalized integral operator	Complex formulation	Integral transforms
Weyl	Infinite interval problems	Limited physical interpretation	Fourier analysis
Saigo	Generalizes RL and EK	Hypergeometric kernel	Special functions
Marichev-Saigo-Maeda (MSMO)	Highly generalized	High computational complexity	Hypergeometric functions

3 Fundamental of Fractional Calculus

$\mathcal{FC}$ and Quantum calculus (QC) have become interesting topics for researchers in this field, as they have proven very useful and applicable. Considering this interest and its applications, (Saxena et al. 2019) articulated a formulation for classical quantum mechanics based on $\mathcal{FC}$. The formulation indicates that the quantum system can be represented by the Schrödinger equation, which is as follows:

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial t^2} + V(t) \right) \psi(t) = E\psi(t).$$

When the potential is exactly known, the problem becomes easier to study. However, exact potentials

are difficult to find in many physical systems. Therefore, the authors introduced conformable $\mathcal{FC}$ to obtain better theoretical results and improve experimental observation. The conformable fractional derivative is defined as:

$$D_{\alpha}^t f(t) = |t|^{1-\varrho} \frac{\partial f(t)}{\partial t}, \quad (\varrho < 1).$$

The authors extend the classical wave equation using conformal fractional derivatives and conclude that the fractional parameter plays an important role in altering wave propagation, wavelength, period, and dynamics. The paper develops fractional quantum mechanics through the generalized Schrödinger equation:

$$i\hbar_{\alpha} D_{\tau}^{\alpha} \psi = \left[-\frac{\hbar_{\varrho}^2}{2m_{\varrho}} (D_t^{\varrho})^2 + V_{\alpha}(t) \right] \psi.$$

Overall, the paper concludes that conformable $\mathcal{FC}$ provides a generalized and flexible approach to quantum mechanics. The deformation parameter $\aleph$ alters key quantum properties, including wave propagation, energy spectra, probability distributions, density of states, and uncertainty relations. The study suggests that fractional quantum mechanics could become a useful mathematical tool for describing complex quantum systems and achieving better agreement between theoretical models and experimental results.

Reaction-diffusion equations are important mathematical models used to represent diffusion, transport phenomena, and reaction processes in the sciences and population dynamics. These equations explain wave propagation, pattern formation, tumor growth, and migration of biological species. A general approach to solving fractional reaction-diffusion equations was established by (Mathai et al. 2006)

using $\mathcal{CFD}$ along with the Laplace transform ($\mathcal{LT}$) (Bekela et al. 2020) and the Fourier transform ($\mathcal{FT}$) (Cooley et al. 1965). The generalized reaction-diffusion equation considered in the study is,

$${}_0D_v^\rho N(u, v) + c {}_0D_v^\nu N(u, v) + d {}_0D_v^\gamma N(u, v) \\ = \varepsilon {}^2_{-\infty}D_u^\xi N(u, v) + \xi^2 N(u, v) + \psi(u, v).$$

The solutions were obtained in terms of the generalized Mittag-Leffler function ($\mathcal{MLF}$) (Prabhakar et al. 1971), which is defined as:

$$E_{\rho, \nu}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\rho k + \nu) k!}, \quad (13)$$

which plays an important role in $\mathcal{FC}$. The study concludes that fractional parameters strongly affect diffusion and wave-propagation behavior in complex systems. The applications of the two-dimensional Laplace transform together with $\mathcal{MSMO}$ and incomplete hypergeometric functions (Srivastava et al. 2019) were investigated by (Khan et al. 2021). In a related study, Solution of generalized fractional partial differential equations with $\mathcal{HFD}$ and space-fractional generalized Laplace operators that arise in quantum mechanics were studied by (Purohit et al. 2013). The authors used $\mathcal{LT}$ and $\mathcal{FT}$ to obtain analytics solutions, and expressed them in terms of the Fox-H function ($\mathcal{FHF}$) (Fox et al. 1961), which is defined as:

$$H_{p, q}^{m, n}(z) = \frac{1}{2\pi i} \int_L \phi(u) z^{-u} du. \quad (14)$$

Moreover, some special cases are derived, including solutions of one-dimensional non homogeneous fractional equations that appear in quantum mechanics. A new class of Quantum-Fractal-Fractional Operator (QFFOs), based on the combination of

fractal-fractional calculus with $\mathcal{QC}$ was introduced by (Attiya et al. 2025). The classical fractal-fractional operators were generalized to the complex plane using Jackson's q-derivative and q-gamma function. The generalized QFFO is defined as:

$$Q_q^{\sigma, \eta} f(\xi) = \xi + \sum_{n=2}^{\infty} a_n (K_n)_q \xi^n,$$

where $(K_n)_q$ denotes the q-fractal-fractional coefficients obtained through the q-gamma function. The quantum versions of the $\mathcal{RLO}$ and $\mathcal{CFD}$ operators were also introduced in the study. Using these operators, they introduced a new family of analytic and univalent functions on the unit disk. By applying the differential subordination technique, the authors derived sufficient conditions for the operators to be, in turn, univalent, convex, and bounded. Applications involving special functions, such as the Bessel function ($\mathcal{BF}$) and the generalized hypergeometric function ($\mathcal{GHF}$) (Watson et al. 1996) were also discussed in the article. In general, the work established a new connection among $\mathcal{FC}$, fractal analysis, and $\mathcal{QC}$ in geometric function theory. The image product of multi-variable H-functions (Srivastava et al. 1976) and the generalized class of polynomials under $\mathcal{SO}$ are derived by (Agarwal et al. 2012). They also derived special cases like $\mathcal{RL}$ and $\mathcal{EK}$ fractional integral operators, together with some special functions such as the generalized Wright hypergeometric function ($\mathcal{GWHF}$) (Wright et al. 1935), $\mathcal{MLF}$ and the Whittaker function ($\mathcal{WF}$) (Whittaker et al. 1927). The operator $\mathcal{SO}$ was applied (Chandak et al. 2022) to derive fractional integral and differential formulas for the generalized V-function (Mathai et al. 1978) and included various functions like the exponential function, $\mathcal{MLF}$, $\mathcal{BF}$, Struve function (Watson et

al. 1996), Lommel function (Zullo et al. 2025) and hypergeometric functions as special cases. The authors also obtained the results for $\mathcal{RL}$ and $\mathcal{EK}$ fractional calculus as special cases and used the Beta transform ($\mathcal{BT}$) (Chaudhry et al. 1997) and $\mathcal{LT}$ to obtain composition formulas. In a related study various fractional operators was applied (Birhane et al. 2025) to derive the Y-function including $\mathcal{MSMO}$, $\mathcal{SO}$, $\mathcal{EKO}$, $\mathcal{RLO}$ and $\mathcal{WFTO}$ and established several results. The Y-function is defined as:

$$Y_{p,q}^{m,n} \left[\begin{matrix} \{a_p, \mathfrak{A}_p\}_1^p \\ \{b_q, \mathfrak{B}_q\}_1^q \end{matrix} \middle| t; \eta; \varrho \right] = \frac{1}{2\pi i} \int_L \Theta(\varsigma) t^{-\varsigma} e^{-\varsigma \eta} \varsigma^{\varrho} d\varsigma, \quad (15)$$

where

$$\Theta(\varsigma) = \frac{\prod_{j=1}^m \Gamma(b_j - \mathfrak{B}_j \varsigma)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \mathfrak{B}_j \varsigma)} \times \frac{\prod_{j=1}^n \Gamma(1 - a_j + \mathfrak{A}_j \varsigma)}{\prod_{j=n+1}^p \Gamma(a_j - \mathfrak{A}_j \varsigma)}.$$

These results showed that the order of the function increases, but its structure remains unchanged. The (p, q) -extended $\mathcal{BF}$ (Watson et al. 1996) was investigated by (Srivastava et al. 2021) using $\mathcal{MSMO}$ through a simple analytical approach and obtained new formulas. The results are represented through the hypergeometric function and $\mathcal{FWF}$. Subsequently a new image formula for the (p, ς) -extended Gauss hypergeometric function was established by (Chopra et al. 2022) using $\mathcal{SO}$. The (p, ς) -extended Gaussian hypergeometric function

(Parmar et al. 2017) is defined as:

$$F_{p,\varsigma}(q, r, \varsigma; \mathfrak{z}) = \sum_{k=0}^{\infty} (q)_k \frac{B_{\varsigma}(r+k, \varsigma-r; p)}{B(r, \varsigma-r)} \frac{\mathfrak{z}^k}{k!}, \quad (\Re(p) > 0). \quad (16)$$

Also derived special cases for $\mathcal{RLO}$ and $\mathcal{EKO}$; the results are expressed in terms of the Hadamard product of the (p, ς) -extended Gauss hypergeometric function and $\mathcal{FWF}$. They also analyze Jacobi transformation, including special cases such as Gegenbauer and Legendre functions. $\mathcal{CFD}$ and $\mathcal{RLO}$ of the S-function (Sharma et al. 2014) were investigated by (Islam et al. 2025). They derived a distinctiveness theorem for the S-function in terms of $\mathcal{GWHF}$ and also obtained results for a general class of polynomials. $\mathcal{FC}$ formulas were developed by combining an incomplete $\mathfrak{N}$ -functions (Saxena et al. 2006) with a class of polynomials as presented by (Bhatter et al. 2024). The results are presented in a simple, clear form, making them useful for constructing various types of fractional operators. Similarly $\mathcal{SO}$ was employed by (Debalkie et al. 2026) to develop fractional integral and derivative formulas for the generalized function ${}_pF_q$ (Luke et al. 1969) and expressed results in terms of $\mathcal{GWHF}$. They also derived composition formulas using $\mathcal{BT}$ and $\mathcal{LT}$. They explained the theoretical importance of these results and discussed their possible applications in related research fields. Fractional integral formulas for the (p, q) -extended modified Bessel function of the second kind were established using $\mathcal{MSMO}$ by (Chopra et al. 2023). In addition, related results were obtained for $\mathcal{SO}$, $\mathcal{RLO}$, and $\mathcal{EKO}$. The results were represented using the original function and $\mathcal{FWF}$ via the Hadamard product. Several new generalizations involving convex func-

tions, reverse Minkowski inequalities, and reverse Hölder inequalities through the study of QC and FC using the Quantum Hahn operator by (Rashid et al. 2020). They established various fractional quantum integral inequalities by employing the quantum shift operator,

$$\nu_{\varphi}^q(\varsigma) = q\varsigma + (1 - q)\nu,$$

and the Hahn difference operator;

$$D_{q,\varpi}h(\varsigma) = \frac{h(q\varsigma + \varpi) - h(\varsigma)}{\varsigma(q - 1) + \varpi}.$$

The authors showed that, for the limiting case $\lambda = 1$, $q \rightarrow 1$, $\varpi = 0$ The results obtained reduce to classical integral inequalities. This demonstrates that the proposed fractional quantum calculus approach is more effective than classical calculus. Furthermore, an application involving the $\mathcal{RL}$ -type (q, ϖ) -derivative was presented in a boundary value problem to demonstrate the usefulness and efficiency of the proposed operator. The incomplete $\mathfrak{N}$ -functions were studied and expressed in the form of Mellin-Barnes contour integral using incomplete Gamma functions (Sharma et al. 2024) by (Bansal et al. 2020). They also applied transform techniques such as $\mathcal{LT}$, Mellin transform ($\mathcal{MT}$) (Flajolet et al. 1995) and the Sumudu transform ($\mathcal{ST}$) (Watugala et al. 1993) and obtained new integral formulas with the help of $\mathcal{MSMO}$. The paper also showed that the incomplete $\mathfrak{N}$ -functions are generalization of special functions such as the $\mathcal{FHF}$ and the I -function (Sharma et al. 1992). Finally, it was applied to a blood glucose supply model, where glucose variation was represented by the differential equation:

$$\frac{dZ}{dx} = -\zeta(Z - k),$$

where Z represents the glucose quantity and k denotes the supply rate. Thus, the incomplete $\mathfrak{N}$ -functions was applied in glucose supply analysis.

4 Recent Developments in Fractional Calculus

Fractional calculus advanced significantly during the period 2023-2025, spurred by the growing demand for more flexible mathematical models capable of addressing memory-dependent and non-local phenomena. Recent research has focused on extending classical fractional operators to more generalized formulations with better analytical properties and broader applicability. Variable-order operators, non-singular kernel derivatives, proportional fractional operators, and efficient computational algorithms for solving fractional differential equations have all received special consideration. These advancements have expanded the scope of fractional calculus in engineering, physics, biology, finance, and data-driven modelling.

4.1 Generalized fractional operator

Generalized fractional operators are the advanced versions of classical fractional derivatives and ~~and~~ the idea of Riemann-Liouville and Caputo operators (Garrappa et al. 2025, Luchko et al. 2024a, Ortigueira et al. 2024). They provide more flexibility in modelling systems with memory and hereditary effects by including various kernel functions. Examples include the Hilfer, Katugampola, Prabhakar, Saigo and Marichev-Saigo-Maeda fractional operators.

4.2 Variable-order fractional operator A variable-order fractional operator is a fractional operator of variable order (Valério et al. 2023, Khey-bari et al. 2024, Valarmathi et al. 2023, Luchko et

al. 2024b) . The order is varied with respect to time, space or other independent variables. This property allows for a more realistic description of system-changing memory effects than constant-order models. Variable-order operators improve modelling accuracy and are widely used in viscoelastic materials, diffusion processes, control systems, and biomedical engineering, where system dynamics change with time.

4.3 Non-singular kernel operator

Non-singular kernel operators have been introduced to remove the singularity in classical fractional derivatives by replacing the power law kernel with smooth kernels such as exponential or Mittag-Leffler functions (Fernandez et al. 2023, Odibat et al. 2023, Sabatier et al. 2024, Ahmed et al. 2026). This modification improves numerical stability, computational efficiency and representation of fading memory effects. Important examples are the Caputo-Fabrizio and Atangana-Baleanu derivatives, which have been successful in heat transfer, diffusion, fluid mechanics and biological modelling.

4.4 Proportional fractional operator

The proportional fractional operator is a newly developed fractional operator that combines proportional differentiation with fractional-order calculus (Sudsutad et al. 2024, Wang et al. 2024a, Demir et al. 2024). It allows a larger freedom to model systems with proportional growth or decay but retains memory effects. This operator has been applied in control systems, signal processing, electrical circuits and population dynamics because of its better mathematical representation and computational accuracy.

4.5 Emerging Applications and Computational Trends

(Dozva et al. 2026) Fractional calculus is effectively used to model complex systems that exhibit memory and non-local effects. Recent studies combine traditional numerical methods with artificial intelligence techniques, such as Artificial Neural Networks (ANNs), Deep Neural Networks (DNNs), and physics-informed Neural Networks (PINNs), to obtain more accurate and faster solutions of fractional differential equations. These methods are successfully applied in areas such as biomedical engineering, epidemic modelling, finance, fluid mechanics, image processing, and control systems.

5 Application of Fractional Calculus

Fractional calculus has evolved as an effective mathematical framework for studying systems with memory, heredity, and nonlocal properties. Unlike conventional integer-order calculus, fractional derivatives represent a system's history, making them ideal for modelling complex dynamic processes in science and engineering. This capacity has considerably expanded the use of fractional calculus in a variety of study domains, including physics, engineering, biological sciences, economics, and signal processing. The ongoing development of generalized fractional operators has expanded the application of fractional models by increasing flexibility and improving agreement with experimental results.

5.1 Biological and Medical Sciences The intrinsic memory quality of fractional derivatives makes them ideal for biological and biomedical models (Ghezal et al. 2025, Nisar et al. 2024, Vieira

et al. 2023a). Fractional differential equations have been frequently used to study cancer development, epidemic transmission, neuronal dynamics, pharmacokinetics, drug delivery devices, and population dynamics. Compared with standard mathematical models, fractional models provide a more accurate picture of genetic effects and long-term biological behaviour. As a result, they frequently show higher agreement with experimental and clinical findings.

5.2 Physics and Engineering

Fractional calculus is widely used in heat conduction, anomalous diffusion, wave propagation, viscoelasticity and fluid dynamics. The memory-dependent structure of fractional derivatives allows for a more precise description of physical processes than standard integer-order models (Kaur et al. 2025, Krejcar et al. 2026, Singla et al. 2025). In engineering, fractional-order controllers have outperformed standard PID controllers in robustness, stability, and performance, especially in complex dynamic systems.

5.3 Finance and Economics

In financial mathematics, fractional calculus has also drawn a lot of interest for modeling stochastic processes, long-range dependence, market volatility and option pricing (Cheow et al. 2024, Agarwal et al. 2024a, Rahman et al. 2025). Fractional stochastic models provide more modeling flexibility than conventional Brownian-motion-based methods since financial markets often display memory and persistence. These models have enhanced understanding of intricate financial dynamics and risk assessment.

5.4 Signal Processing and Artificial Intelligence

Fractional calculus is now used in signal processing, picture enhancement, machine learning, and artificial intelligence, thanks to recent developments. Because fractional operators better retain accurate data, they have been effectively used for edge detection, image denoising, feature extraction, and pattern recognition (Ghanbari et al. 2024, Yavuz et al. 2025, Joshi et al. 2023, Agarwal et al. 2024b). Additionally, fractional differential equations have recently been combined with Physics-Informed Neural Networks (PINNs) and other machine learning approaches, offering effective computational frameworks for resolving challenging nonlinear and high-dimensional problems.

6 Numerical Methods and Computational Techniques

Fractional differential equations (FDEs) are extremely challenging to solve analytically due to their non-local and memory-dependent nature. As a result, a wide range of analytical, semi-analytical, and numerical techniques have been developed to yield approximation solutions with acceptable accuracy and computing efficiency. The proper method is identified by the problem's complexity, fractional operator type, and requisite level of accuracy.

The Adomian Decomposition Method (ADM) (Hussain et al. 2024, Lan et al. 2023), Homotopy Analysis Method (HAM) (Gao et al. 2025, Alqahtani et al. 2024, Xue et al. 2025) Homotopy Perturbation Method (HPM) (Farhood et al. 2023, Liu et al. 2023, Fareed et al. 2025) and Variational Iteration Method (VIM) (Pazouki et al. 2023, Khan et al. 2024, Singh et al. 2023) are all popular

semi-analytical techniques. These methods are especially useful for getting approximate solutions to linear and nonlinear fractional differential equations without significant discrete computation.

Numerical methods including the Finite Difference Method (Wang et al. 2023, Wang et al. 2024, Gao et al. 2024), Finite Element Method (FEM) (Kumar et al. 2024, Ramadan et al. 2025, Izadi et al. 2025), Spectral Methods, and Predictor-Corrector algorithms are often used for increasingly complicated fractional models. These techniques offer reliable and precise solutions for a variety of physical, biological, and engineering applications. Physics-Informed Neural Networks (PINNs) (Sivalingam et al. 2024, Hu et al. 2024, Nuugulu et al. 2025) and other machine learning-based computational techniques have also been introduced recently (2023–2025), and they have shown promising performance in addressing inverse fractional and high-dimensional issues. However, computational cost, convergence, and parameter estimation remain significant issues that spur further study.

Table-2 Comparison of Numerical Methods

Method	Main Feature	Application
ADM	Semi-analytical	Nonlinear FDEs
HAM	Adjustable convergence	Fractional models
HPM	Simple implementation	Engineering problems
FDM	Numerical discretization	Diffusion equations
Predictor-Corrector	Time-stepping method	Initial value problems
PINNs	AI-based approach	High-dimensional FDEs

Table-2 summarizes the computational techniques commonly used to solve fractional differential equations. While numerical techniques like the Predictor-Corrector Method and Finite Difference Method are frequently used for complex nonlinear models, semi-analytical techniques like ADM and HAM are appropriate for getting approximate analytical solutions. Physics-Informed Neural Networks (PINNs) have recently become a promising computational technique for solving high-dimensional fractional problems.

7 Research gap

Fractional calculus has made tremendous progress, yet there are still a number of critical research gaps. First, there is still a lack of thorough comparisons between classical and generalized fractional operators, which makes it challenging to determine which operator is best for a given application. Second, while generalized and non-singular kernel operators have attracted considerable interest, their experimental validation and physical interpretation are still lacking. Third, although variable-order and proportional fractional operators have provided encouraging outcomes, further research is needed to fully understand their theoretical characteristics and numerical applications. Additionally, there is ongoing work to create effective computing methods for solving large-scale nonlinear fractional differential equations. These gaps show that comprehensive theoretical, numerical, and application-focused research is required.

8 Future Research Direction

Future research in fractional calculus should focus on establishing unifying frameworks that create explicit links among distinct fractional operators. Further work is also necessary to improve physical

comprehension of generalized operators and to create effective numerical techniques for large-scale fractional differential equations. Emerging research fields include variable-order fractional models, fractional optimal control, artificial intelligence-based computational approaches such as Physics-Informed Neural Networks (PINNs), quantum fractional models, biomedical applications, and data-driven modelling. The advancement of the area will also be supported by reliable computational tools. Computational tools and experimental validation of fractional models will greatly aid the advancement of the area. This will involve developing reliable computational tools and experimentally validating these fractional models.

9 Conclusion

Fractional calculus has developed from a theoretical branch of mathematics into an important tool for modelling complex phenomena that exhibit memory and hereditary characteristics. The increasing need to describe non-local behaviour in real-world systems has encouraged the development of a wide variety of fractional operators, each possessing unique mathematical properties and application potential. This review has examined fundamental and generalized fractional operators, including the Riemann–Liouville, Caputo, Hilfer, Erdélyi–Kober, Weyl, Saigo, and Marichev–Saigo–Maeda operators, and discussed their theoretical foundations and practical significance.

A comparative study of these operators indicates that every fractional operator has its own advantages and limitations. Classical operators remain mathematically well-established and are widely used in theoretical investigations, whereas recently developed generalized, variable-order, proportional,

and non-singular kernel operators provide greater flexibility for modelling complex physical and engineering systems. Consequently, the choice of an appropriate operator should be based on the characteristics of the problem under consideration rather than adopting a single operator for all applications. This review has also highlighted the broad applicability of fractional calculus in physics, engineering, biological sciences, finance, signal processing, and other interdisciplinary fields. In addition, recent advances in numerical methods and computational techniques have considerably improved the practical implementation of fractional differential equations. Emerging computational approaches, including machine learning and Physics-Informed Neural Networks (PINNs), further demonstrate the growing role of fractional calculus in modern scientific computing.

Despite these achievements, several important issues remain unresolved. The physical interpretation of many generalized fractional operators is still an active area of investigation, and further work is needed to establish unified theoretical frameworks and reliable criteria for selecting suitable operators for different classes of problems. Improving computational efficiency, developing robust numerical algorithms, and validating fractional models through experimental studies also remain important research challenges.

In summary, fractional calculus continues to evolve as a dynamic and rapidly expanding research area. Ongoing developments in generalized operators, computational techniques, and interdisciplinary applications are expected to strengthen both its mathematical foundations and its practical relevance. Future research is expected to further expand the scope of fractional calculus and promote its appli-

cation to increasingly complex problems arising in science, engineering, and technology.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

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References

P. Agarwal, (2012). Further Results on Fractional Calculus of Saigo Operators. *Appl. Appl. Math.*, **7(2)**, 585–594.

Agarwal, P., Martínez, L. V., Lenzi E. K., (2024). Recent Trends in Fractional Calculus and Its Applications. *Elsevier*.

Agarwal, P., Cattani, C., Momani. S., (2024). Fractional Differential Equations: Theoretical Aspects and Applications. *Elsevier*.

Ahmed, S.A., Al-Safi, M.G.S., Shah. R., (2026). Sumudu decomposition method with non-singular kernel operators for solving time fractional Whitham-Broer-Kaup equations. *Bound Value Probl.* **33**.

Alqahtani, A. M., Prasad J. G., (2024). Solution of local fractional generalized coupled Korteweg–de Vries (cKdV) equation using local fractional homotopy analysis method and Adomian decomposition method. *Applied Mathematics in Science and Engineering*. **32(1)**.

Attiya, A. A., Ibrahim, R. W., Hakami, A. H., Cho, N. E., Yassen, M. F., (2025). Quantum–Fracta Fractional Operator in a Complex Domain. *Axioms*, **14**, 57.

Bansal, M. K., Kumar, D., Nisar, K. S., Singh, J., (2020). Certain fractional calculus and integral transform results of incomplete $\mathfrak{N}$ -functions with applications. *Math. Meth. Appl. Sci.*, **43**, 5602–5614.

Bekela, A. S., Belachew, M. T., Wole, G. A., (2020). A numerical method using Laplace-like transform and variational theory for solving time-fractional nonlinear partial differential equations with proportional delay. *Advances in Difference Equations*, **1**, 586.

Bhatter, N., Purohit, S. D., Nisar, K. S., Munjam, S. R., (2024). Some fractional calculus findings associated with the product of incomplete $\mathfrak{N}$ -function and Srivastava polynomials. *International Journal of Mathematics and Computer in Engineering*, **2(1)**, 97–116.

Birhane, E., Suthar, D. L., (2025). Images for the Y-Function via Marichev–Saigo–Maeda Fractional Integration and Differentiation Operators. *International Journal of Mathematics and Mathematical Sciences*, 4523398, 16 pages.

Canavati, J. A., (1987). The Riemann–Liouville integral. *Nieuw Archief Voor Wiskunde*, **5(1)**, 53–75.

Caputo, M., (1967). Linear models of dissipation whose Q is almost frequency independent–II. *Geophysical Journal International*, **13(5)**, 529–539.

- Chandak, S., Alaria, A. , Shimelis, B., (2022). Composition Formula for Saigo Fractional Integral Operator Associated with V-Function. *Hindawi Journal of Mathematics*, 2174708, 11 pages.
- Chaudhry, M. A., Qadir, A., Rafique, M., Zubair, S. M., (1997). Extension of Euler's Beta function. *Journal of Computational and Applied Mathematics*, **78**(1), 19–32.
- Cheow, Y. H., Ng, K. H., Phang C., (2024). The application of fractional calculus in economic growth modelling: An approach based on regression analysis. *Heliyon (Elsevier)*, **10**(15), e35379.
- Chopra, P., Gupta, M., Modi, K., (2023). Fractional integration and differentiation of the function (p,q)-extended modified Bessel function of the second kind and integral transforms. *Communications of the Korean Mathematics Society*, **38**(3), 755–772.
- Chopra, P., Gupta, M., Modi, K., (2022). Certain image formulas of (p, q)-extended Gauss' hypergeometric function and related Jacobi transforms. *Commun. Korean Math. Soc.*, **37**(4), 1055–1072.
- Cooley, J. W., Tukey, J. W., (1965). An algorithm for the machine calculation of complex Fourier series. *Mathematics of Computation*, **19**(90), 297–301.
- Debalkie, D. L. Suthar, (2026). Composition Formula for Saigo Fractional Calculus Operator on pRq Function. *International Journal of Mathematics and Mathematical Sciences*, 9934846, 19 pages.
- Demir, İ., Tunç. T., (2024). New midpoint-type inequalities in the context of the proportional Caputo-hybrid operator. *J Inequal Appl.* **2**.
- Dozva, R., Goqo, S. P., Oloniju. S. D., (2026). A Critical Review of Computational Methods for Fractional Differential Equations: Bridging Classical Discretizations and Learning-Based Approximations. *Arch Computat Methods Eng.* **33**, 7571–7594.
- Fareed, A. F., Mohamed, E. A., Aly, M., Semary, M. S., (2025). A Novel Fractional Integral Transform-Based Homotopy Perturbation Method for Some Nonlinear Differential Systems. *Fractal Fract.* **9** , 212.
- Farhood, A. K., Mohammed. O. H., (2023). Homotopy perturbation method for solving time-fractional nonlinear variable-order delay partial differential equations. *Partial Differential Equations in Applied Mathematics*, **7**, 100513.
- Fernandez, A., Al-Refai. M., (2023). A rigorous analysis of integro-differential operators with non-singular kernels. *Fractal and Fractional.* **7**(3), 213.
- Flajolet, P., Gourdon, X., Dumas, P., (1995). Mellin transforms and asymptotics, harmonic sums. *Theoretical Computer Science*, **144**(1–2), 3–58.
- Fox, C., (1961). The G and H Functions as Symmetrical Fourier Kernels. *Transactions of the American Mathematical Society*, **98**(3), 395–429.
- Gao, G., Sun, Z., Zhang H., (2024). An efficient finite difference scheme for nonlinear time-fractional reaction–diffusion equations. *Journal of Computational Physics.* 505, 112913.

- Gao, Q., Li, J. H., Wang, W., Wang, X., Hwang, H. J., Lee, Y. H., (2025). Homotopy analysis method for linear and nonlinear stochastic problems. *Probabilistic Engineering Mechanics* . **79**.
- Garrappa, R., Górska, K., Kaslik, E., (2025). Generalized fractional operators do not preserve periodicity. *Fract Calc Appl Anal*. **28**, 1681–1705.
- Ghanbari, B., (2024). Fractional Calculus: Bridging Theory with Computational and Contemporary Advances. *Elsevier*.
- Ghezal A., and Attia. N., (2025). ν -Fractional Stochastic Models for Simulating Tumor Growth and Chemical Diffusion in Biological Tissues. *Fractal Fract*. **9**, 258.
- Herrmann, R., (2014). Towards a geometric interpretation of generalized fractional integrals-Erdelyi-Kober type integrals on $\mathbb{R}^N$, as an example. *Fractional Calculus and Applied Analysis*, **17(2)**, 361–370.
- Hilfer, R., (2000). Applications of Fractional Calculus in Physics. *World Scientific Publishing, Singapore*.
- Hu, Z., Kawaguchi, K., Zhang, Z., Em Karniadakis, G., (2024). Tackling the curse of dimensionality in fractional and tempered fractional PDEs with physics-informed neural networks. *Computer Methods in Applied Mechanics and Engineering*. **432**, 117448.
- Hussain, S., Arora, G., Kumar, R., (2024). Semi-analytical methods for solving non-linear differential equations: A review. *Journal of Mathematical Analysis and Applications*. **531(1)**, 127821.
- Islam, M.S., Kumari, M., Daiya, J., Mehla, S., (2025). Caputo-type Marichev-Saigo-Maeda fractional derivative and general class polynomial associated with S-function. *International Journal of Applied Mathematics*, 38 No. 11s.
- Izadi, M., Srivastava, H. M., Kamandar. M., (2025). The LDG finite-element method for multi-order fractional differential equations: Applications to circuit equations. *Fractal and Fractional*. **9(4)**, 230.
- Joshi, M., Bhosale, S., Vyawahare. V. A., (2023). A Survey of Fractional Calculus Applications in Artificial Neural Networks. *Artificial Intelligence Review (Springer Nature)*.
- Kaur, L., Kumar. P., (2025). Recent Developments in Fractional Calculus: Theory, Applications, and Numerical Simulations. *Springer Nature Switzerland*.
- Khan Y., Khan, A., Shaeel, M., Akgül, A., (2021). Two Dimensional Laplace Transform Coupled with the Marichev-Saigo-Maeda Integral Operator and the Generalized Incomplete Hypergeometric Function. *Symmetry*, **13**, 2420.
- Khan, A., Ali, M., Shah. K., (2024). A modified variational iteration method for nonlinear fractional differential equations with Caputo derivatives. *Advances in Difference Equations. Springer Nature*.
- Kheybari, S., Alizadeh, F., Darvishi, M.T., Hosseini, K., Hincal, E., (2024). A Novel Semi-Analytical Scheme to Deal with Fractional Partial Differential Equations of Variable-Order. *Fractal and Fractional*. **8(7)**, 425.

- Krejcar, O., Namazi, H., (2026). Review of the applications of different fractional models in engineering. *Applied Mathematical Modelling*. **153**, 116623.
- Kumar, L., Sista, S. G., and Konijeti, S., (2024). Finite element analysis of time-fractional integro-differential equation of Kirchhoff type for non-homogeneous materials, *Math. Meth. Appl. Sci* **47**, 2120–2153.
- Lacroix, S. F., (1819). *Traite du Calcul Differentiel et du Calcul Integral*, II ed., Courcier, Paris .
- Lan, K., Webb. J. R. L., (2023). A new Bihari inequality and initial value problems of first-order fractional differential equations. *Fractional Calculus and Applied Analysis* . **26(3)**, 962–988.
- Liu, J., Nadeem, M., Iambor. L. F., (2023). Application of Yang homotopy perturbation transform approach for solving multi-dimensional diffusion problems with time-fractional derivatives. *Scientific Reports*. **13**, 21855.
- Luchko. Y., (2024). General fractional calculus operators with the Sonin kernels and some of their applications. *IFAC-PapersOnLine*. *58(12)*, 302–311.
- Luchko, Y., Kiskinov, H., Zahariev. A., (2024). Variable-Order Fractional Linear Systems with Distributed Delays—Existence, Uniqueness and Integral Representation of the Solutions. *Fractal and Fractional*. **8(3)**, 156.
- Luke, Y. L., (1969). *The Special Functions and Their Approximations*. *Academic Press*.
- Mathai, A. M., Saxena, R. K., (1978). *The H-Function with Applications in Statistics and Other Disciplines*. *Wiley Eastern*.
- Mathai, A. M., Saxena, R. K., Haubold, H. J., (2006). Fractional reaction-diffusion equations. *Astrophysics and Space Science*, **305**, 289–296.
- Nisar, K. S., Farman, M., Abdel-Aty, M., Ravichandran, C., (2024). A review of fractional order epidemic models for life sciences problems: Past, present and future. *Alexandria Engineering Journal*. **95**, 283–305.
- Nuugulu, S. M., Patidar, K. C., Tarla. D. T., (2025). A physics informed neural network approach for solving time fractional Black–Scholes partial differential equations. *Optimization and Engineering*. **26**, 2419–2448.
- Odibat, Z., Baleanu. D., (2023). A new fractional derivative operator with generalized cardinal sine kernel: Numerical simulation. *Mathematics and Computers in Simulation*. **212**, 224–233.
- Ortigueira. M. D., (2024). Searching for Sonin kernels. *Fractional Calculus and Applied Analysis*. **27**, 2219–2247.
- Parmar, R. K., Chopra, P., Paris, R. B., (2017). On an extension of extended beta and hypergeometric functions. *J. Class. Anal.*, **11(2)**, 91–106.
- Pazouki, M., Ghanbari, B., Babolian, E., (2023). Comparison of variational iteration and Adomian decomposition methods to solve growth, aggregation and aggregation-breakage equations. *Journal of Computational Science*, **67**, 101973.
- Prabhakar, T. R., (1971). A singular integral equation with a generalized Mittag-Leffler function

in the kernel. *Yokohama Mathematical Journal*, **19**, 7–15.

Purohit, S. D., (2013). Solutions of Fractional Partial Differential Equations of Quantum Mechanics. *Adv. Appl. Math. Mech.*, **5**, 639–651.

Rahman, M., Asif, M., Shah, N. A., (2025). A new perspective on order-theoretic contractive mappings with applications in financial markets and economic growth via fractional differential equations. *Boundary Value Problems (Springer Nature)*, **175**.

Ramadan, M., Samy, H., Hanafy, I., Adel, W., (2025). Petrov–Galerkin finite element method for solving the time-fractional Rosenau–Hyman equation. *Journal of Umm Al-Qura University for Applied Sciences*.

Rashid, S., Hammouch, Z., Ashraf, R., (2020). New quantum estimates in the setting of fractional calculus theory. *Adv. Differ. Equ.*, **383**.

Sabatier, J., Farges, C., (2024). Time-domain fractional behaviour modelling with rational non-singular kernels. *Axioms*. **13**(2), 99.

Saigo, M., (1978). A Remark on Integral Operators involving the Gauss hypergeometric functions. *Math. Rep. College General Ed. Kyushu Univ.*, **11**, 135–143.

Samko, S. G., Kilbas, A. A., Marichev, O. I., (1993). Fractional Integrals and Derivatives: Theory and Applications. *Gordon and Breach, Yverdon*.

Saxena, R. K., Mathai, A. M., Haubold, H.J., (2006). On generalized fractional kinetic equations associated with the Aleph-function. *Applied Mathematics and Computation*. **181**, 288–301.

Saxena, R. K., Chung, W. S., Zare, S., Hassanabadi, H., Maghsoodi, E., (2019). The effect of fractional calculus on the formation of quantum-mechanical operators. *Math. Meth. Appl. Sci.*, 1–18.

Sharma, C. K., Al-Musallam, A. A., (1992). A new representation of generalized functions. *International Journal of Mathematics and Mathematical Sciences*, **15**(2), 321–326.

Sharma, C. K., Jain, R., (2014). On the generalized S-function and its applications. *International Journal of Mathematical Analysis*, **8**(9), 425–434.

Sharma, R., Kumar, D., Singh, J., Singh, Y., (2024). On the Certain Properties and Results of Incomplete Generalized Hypergeometric Functions with Matrix Argument. *Journal of Computational Analysis and Applications*, **33**(1A), 430–446.

Singh, J., Kumar, D., Baleanu, D., (2023). Variational iteration method for fractional reaction–diffusion equations arising in applied sciences. *International Journal of Computer Mathematics*.

Singla, K., and Leonenko, N., (2025). On solutions of fractional nonlinear Fokker-Planck equation. *Fract Calc Appl Anal*. **28**, 1094–1105.

Sivalingam, S. M., Kumar, P., Govindaraj, V., (2024). A novel optimization-based physics-informed neural network scheme for solving fractional differential equations. *Engineering with Computers*. **40**, 855–865.

- Srivastava, H.M., Panda, R., (1976). Some bi-lateral generating functions for a class of generalized hypergeometric polynomials. *Journal für die reine und angewandte Mathematik*. **283**, 265–274.
- Srivastava, H. M., Parmar, R. K., Chopra, P., (2012). Some families of the incomplete H-functions and the incomplete hypergeometric functions. *Applied Mathematics and Computation*, **219** , 1819–1827.
- Srivastava, H. M., AbuJarad, E. S. A., Jarad, F., Srivastava, G., AbuJarad, M. H. A., (2021). The Marichev-Saigo-Maeda Fractional-Calculus Operators Involving the (p, q)-Extended Bessel and Bessel-Wright Functions. *Fractal Fract.* **5**, 210.
- Sudsutad, W., Kongson, J., Thaiprayoon, C., (2024). On generalized -Hilfer proportional fractional operator and its applications to the higher-order Cauchy problem. *Bound Value Probl.* **83**.
- Valarmathi, R., Arulprakash, G., (2023). On the variable order fractional calculus of fractal interpolation functions. *Fractional Calculus and Applied Analysis*. **26(1)**.
- Valério, D., Ortigueira, M. D., (2023). Variable-Order Fractional Scale Calculus. *Mathematics*. **11(21)**, 4549.
- Vieira, L. C., Costa, R. S., and Valério, D., (2023). An Overview of Mathematical Modelling in Cancer Research: Fractional Calculus as Modelling Tool. *Fractal Fract.* **7**, 595.
- Wang, Z., Vong, S., (2023). A high-order finite difference method for solving the time-fractional diffusion equation. *Applied Numerical Mathematics*. **191**, 109–126.
- Wang, C. Li, Y., (2024). Finite difference approximation for Caputo fractional partial differential equations: Stability and convergence analysis. *Numerical Algorithms*.
- Wang, H., (2024). Generalized fractional (ρ, κ, ϕ) -proportional Hilfer derivatives and some properties. *Journal of Mathematics* . **17**, 4864945.
- Watson, G. N., (1966). A Treatise on the Theory of Bessel Functions (2nd ed.). *Cambridge University Press*.
- Watugala, G.K., (1993). Sumudu transform: A new integral transform to solve differential equations and control engineering problems. *International Journal of Mathematical Education in Science and Technology*. **24(1)**, 35–43.
- Whittaker, E. T., Watson, G. N., (1927). A Course of Modern Analysis. *Cambridge University Press*.
- Wright, E. M., (1935). The asymptotic expansion of the generalized hypergeometric function. *Journal of the London Mathematical Society*. **10** , 286–293.
- C. Xue, XF. Xu, Zhuang, XN., (2025). Quantum homotopy analysis method with quantum-compatible linearization for nonlinear partial differential equations. *Sci. China Phys. Mech. Astron.* **68**, 104702 .
- Yavuz, M., Singh, D. K., Townley, S., (2025). Recent Developments in Theory and Applications of Fractional Order Systems. *Elsevier* .

Zullo, F., (2025). Lommel functions, Padé approximants and hypergeometric functions. *Ap-*

plied Numerical Mathematics, **209**, 275–284.