



Research Article

Applications of Elzaki Transform to Problems in Physical and Basic Sciences

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Keywords

Elzaki transform,
Differential equations,
Integral transform,
Physical or Basic sciences,
Newton's law of cooling,
Population dynamic.

Abstract

In the present article, we discuss how Elzaki transform are essential tools in various scientific fields for modeling and solving complex problems that arise in real-world scenarios. These problems include population growth and decay, glucose absorption, disease spread, GDP dynamics, Newton's law of cooling, and free-fall motion. The Elzaki transform provides an effective mathematical framework for analyzing these phenomena, enabling accurate modeling of system behavior over time in physical and basic sciences.

Introduction

Many phenomena in physical and basic sciences are effectively modeled using differential equations and exponential functions, which provide more accurate descriptions than ordinary calculus. Integral transforms are widely used to solve the resulting initial value problems in science and engineering (Khalid et al., 2015; Gandolfo, 1997; Acemoglu, 2012; Elzaki, 2012). Typical applications include population dynamics, glucose absorption, free fall in a viscous medium, and Newton's law of cooling. These models help describe growth, decay, stabilization, and other dynamic processes with greater accuracy and provide valuable insights into the behavior of real-world systems (Dayal, 2020; Zill, 2020; Aggarwal et al., 2018; Elzaki, 2011; Oldham and Spanier, 1974; Miller and Ross, 1993).

Elzaki Transform: The Elzaki transform can simplify the solving of differential equations and integral equations more efficiently than the Laplace transform, particularly for certain types of boundary value problems. The initial and final value theorems for the Elzaki transform are straightforward and can be easier to apply in some contexts compared to the Laplace transform. The Elzaki transform may handle functions with singularities differently, which can be advantageous depending on the problem at hand (Khalid et al., 2015). The Elzaki transform of a function is defined as,

$$\Xi[f(u)] = t^2 \int_0^{\infty} f(tu) e^{-u} du = S(t), \quad t \in (k_1, k_2)$$

Elzaki Transform of some function

$f(u)$	$\Xi[f(u)] = S(t)$
1	t^2
u	t^3
u^m	$m! t^{m+2}$
e^{au}	$\frac{t^2}{1-at}$
ue^{au}	$\frac{t^3}{(1-at)^2}$
$u^{a-1} / \Gamma(a)$	t^{a+1}
$\frac{u^{m-1} e^{au}}{(m-1)!}$	$\frac{t^{m+1}}{(1-at)^m}$

Elzaki transform of some derivatives are given as

$$\# \Xi[f'(u)] = \frac{S(t)}{t} - tf'(0) .$$

$$\# \Xi[f''(u)] = \frac{S(t)}{t^2} - f(0) - tf'(0) .$$

$$\# \Xi[f^m(u)] = \frac{S(t)}{t} - \sum_{k=0}^{m-1} t^{2-m+k} f^{(k)}(0) .$$

2. Methodology: In this section, the Elzaki transform is applied for solving some problems like,

2.1. Growth and decay of populations:

Growth and decay of populations are important concepts in ecology, biology, and demography. They describe how the number of individuals in a population increases or decreases over time.

1. Malthusian Growth Model: The Malthusian Growth Model, proposed by Thomas Malthus in the late 18th century, describes how populations grow exponentially when resources are unlimited. It is a fundamental concept in demography and ecology, illustrating the potential for rapid population increase and the eventual constraints imposed by limited resources. The Malthusian Growth Model is a simple yet powerful tool for understanding the potential for exponential population growth in the absence of resource limitations. While it has significant limitations, it lays the groundwork for more nuanced models that incorporate environmental constraints and other real-world factors (Gupta and Mahajan, 2024; Aggarwal, et al., 2018).

Represented by the equation:

$$\frac{dp(\tau)}{d\tau} = \lambda p(\tau) \dots \dots \dots (1),$$

$p(\tau)$ is the population at time τ ,

$p(0) = p_0$ is the initial population size (when time $\tau = 0$),

λ is the intrinsic rate of increase or decrease,

Solution: taking the Elzaki transform of Eq. (1), we get

$$p(t) - t^2 p(0) = \lambda t p(t) ,$$

After simplify the equation,

$$(1 - \lambda t) p(t) = t^2 p(0) ,$$

Put $p(0) = p_0$, we get

$$p(t) = \frac{t^2 p_0}{(1 - \lambda t)} ,$$

Taking inverse Elzaki transform, we get

$$p(\tau) = p_0 e^{\lambda \tau} .$$

e is the base of the natural logarithm.

$p(\tau) = \infty$, if death rate < birth rate, so population approaches ∞ .

For negative value of λ ,

$$\lim_{\tau \rightarrow \infty} p(\tau) = \lim_{\tau \rightarrow \infty} p_0 e^{-\lambda \tau} .$$

$\lim_{\tau \rightarrow \infty} p(\tau) = 0$, here death rate > birth rate, so population approaches 0.

This describe how the number of individuals in a population increases or decreases exponentially.

2. Verhulst model: This model was developed by Pierre-François Verhulst in 1838 and is widely used in ecology and other fields to describe how populations grow in

environments with limited resources. The Verhulst model provides a more realistic description of population growth compared to the Malthusian model, as it considers resource limitations (Gupta, 2020; Elzaki, (2012); Gupta, 2023).

$$\frac{dp(\tau)}{d\tau} = \lambda p(\tau) [1 - \lambda p(\tau)] \quad (2)$$

Solution: we can write Eq. (2) as,

$$\begin{aligned} \frac{dp(\tau)}{d\tau} &= \lambda p(\tau) - \lambda^2 p^2(\tau) , \\ \frac{dp(\tau)}{d\tau} - \lambda p(\tau) &= -\lambda^2 p^2(\tau) , \end{aligned} \quad (2a)$$

Let $Z = \frac{1}{p}$, then

$$-p^2 \frac{dZ(\tau)}{d\tau} = \frac{dp(\tau)}{d\tau} ,$$

Then Eq. (2a) becomes,

$$-p^2 \frac{dZ(\tau)}{d\tau} - \lambda p(\tau) = -\lambda^2 p^2(\tau) ,$$

$$\frac{dZ}{d\tau} + \lambda Z = \lambda^2 ,$$

$$\frac{dZ}{d\tau} = -\lambda Z + \lambda^2 ,$$

Taking Elzaki transform, we have

$$Z(t) - t^2 Z(0) = t (-\lambda Z + \lambda^2 t^2) ,$$

$$Z(t) + \lambda t Z(t) = \lambda^2 t^3 + t^2 Z(0) ,$$

$$Z(t) = \lambda^2 t^3 \frac{1}{(1 + \lambda t)} + \frac{t^2}{p_0(1 + \lambda t)} ,$$

$$Z(t) = \left(\frac{1}{p_0} - \lambda \right) \frac{t^2}{(1 + \lambda t)} + \lambda t^2 , \quad (2b)$$

Taking inverse Elzaki transform of Eq. (2b)

$$Z(\tau) = \left(\frac{1}{p_0} - \lambda \right) e^{-\lambda \tau} + \lambda$$

Therefore,

$$p(\tau) = \frac{1}{\left(\frac{1}{p_0} - \lambda \right) e^{-\lambda \tau} + \lambda} ,$$

$$p(\tau) = \frac{p_0}{(1 - p_0 \lambda) e^{-\lambda \tau} + p_0 \lambda} .$$

When $\lim_{\tau \rightarrow \infty} ,$ then $p(\tau) = \frac{1}{\lambda}$ which is independent of initial population p_0 .

2.2. Glucose immersion by the body:

Glucose absorption and utilization in the body is a complex and vital process that involves several steps, primarily occurring in the digestive system and involving various organs and cells. At $\tau > 0$, we assume that $G(\tau)$ is the units of glucose in the bloodstream. Since the glucose being absorbed by the body is leaving the

bloodstream, so the $\mathcal{G}(\tau)$ satisfies the following equation as [13]

$$\frac{d\mathcal{G}(\tau)}{d\tau} = -\lambda\mathcal{G}(\tau), \quad (3)$$

Taking Elzaki transform of Eq. (3), we get

$$\mathcal{G}(t) - t^2\mathcal{G}(0) = -\lambda t\mathcal{G}(t)$$

We put $\mathcal{G}(0) = \mathcal{G}_0$, the equation become

$$\mathcal{G}(t) = \frac{t^2\mathcal{G}_0}{(1+\lambda t)}$$

taking inverse Elzaki transform, we have

$$\mathcal{G}(\tau) = \mathcal{G}_0 e^{-\lambda\tau}.$$

From above equation it is clear that the units of glucose in the bloodstream fall off exponentially.

Suppose that by the fixed rate of γ unit/sec, the glucose is injected, then $\mathcal{G}(\tau)$ satisfies the following equation as

$$\frac{d\mathcal{G}(\tau)}{d\tau} = -\lambda\mathcal{G}(\tau) + \gamma, \quad (4)$$

here $-\lambda\mathcal{G}(\tau)$ show the immersion of the glucose by the body and γ is due to injection. Also we have $\mathcal{G}(\tau) = \mathcal{G}_0$.

Solution:

$$\mathcal{G}(t) - t^2\mathcal{G}(0) = -\lambda t\mathcal{G}(t) + \gamma t^3$$

$$\mathcal{G}(t) = \frac{t^2\mathcal{G}_0}{(1+\lambda t)} + \frac{\gamma t^3}{(1+\lambda t)},$$

$$\mathcal{G}(t) = \frac{t^2\mathcal{G}_0}{(1+\lambda t)} + \frac{\gamma}{\lambda} \left(t^2 - \frac{t^2}{(1+\lambda t)} \right),$$

$$\mathcal{G}(t) = \frac{t^2}{(1+\lambda t)} \left(\mathcal{G}_0 - \frac{\gamma}{\lambda} \right) + \frac{\gamma}{\lambda} t^2$$

Taking inverse Elzaki transform, we have

$$\mathcal{G}(\tau) = e^{-\lambda\tau} \left(\mathcal{G}_0 - \frac{\gamma}{\lambda} \right) + \frac{\gamma}{\lambda}.$$

2.3. Spread of epidemics: The spread of the epidemics model is given by the equation as

$$\frac{d\mathbf{I}(\tau)}{d\tau} = \lambda\mathbf{I}(\tau)[N - \mathbf{I}(\tau)], \quad (5)$$

where λ is a positive constant. $\mathbf{I}(\tau)$ is the infected population at any time T , N is the total susceptible population and $[N - \mathbf{I}(\tau)]$ is the susceptible people, but still not infected. At initial time $\tau = 0$, the population of infected people is $\mathbf{I}_0$.

Solution: The equation (5) can be written as

$$\frac{d\mathbf{I}}{d\tau} = N\lambda\mathbf{I} - \lambda\mathbf{I}^2,$$

$$\frac{d\mathbf{I}}{d\tau} - N\lambda\mathbf{I} = -\lambda\mathbf{I}^2, \quad (5a)$$

Let $Z = \frac{1}{I}$ then, $-I^2 \frac{dZ(\tau)}{d\tau} = \frac{dI(\tau)}{d\tau}$, then Eq. (5a) becomes

$$\begin{aligned} -I^2 \frac{dZ(\tau)}{d\tau} - N\lambda I &= -\lambda I^2 \\ \frac{dZ}{d\tau} + N\lambda Z &= \lambda, \\ \frac{dI}{d\tau} &= -N\lambda I + \lambda I \end{aligned} \quad (5b)$$

Taking Elzaki transform to Eq.(5b), we have

$$\begin{aligned} Z(t) - t^2 Z(0) &= -N\lambda t Z(t) + \lambda t^3, \\ Z(t) + N\lambda t Z(t) &= \frac{t^2}{I_0} + \lambda t^3, \\ Z(t) &= \frac{t^2}{I_0(1+N\lambda t)} + \frac{\lambda t^3}{(1+N\lambda t)}, \\ Z(t) &= \frac{t^2}{I_0(1+N\lambda t)} + \frac{1}{N} \left[t^2 - \frac{t^2}{(1+N\lambda t)} \right], \\ Z(t) &= \left[\frac{1}{I_0} - \frac{1}{N} \right] \frac{t^2}{(1+N\lambda t)} + \frac{1}{N} t^2, \end{aligned}$$

Taking inverse Elzaki transform, we have

$$Z(\tau) = \left[\frac{1}{I_0} - \frac{1}{N} \right] e^{-N\lambda \tau} + \frac{1}{N}$$

Therefore,

$$I(\tau) = \frac{1}{\left[\frac{1}{I_0} - \frac{1}{N} \right] e^{-N\lambda \tau} + \frac{1}{N}}.$$

$\lim_{\tau \rightarrow \infty} \tau$ approaches infinity, $I(\tau) = N$ which show that all susceptible people eventually become infected.

2.4. Changes in GDP: Gross Domestic Product (GDP) is a key indicator used to gauge the health of a country's economy. It represents the total monetary value of all goods and services produced over a specific time period within a nation's borders. Investors use GDP growth rates to make informed decisions about investments in particular countries or sectors (Gandolfo, 1997; Acemoglu, 2012).

Now, we know that changes in GDP with respect to time are directly proportional to the current GDP which is given by equation.

$$\frac{d\chi(\tau)}{d\tau} = \lambda \chi(\tau), \quad (6)$$

where λ is a positive/negative constant and at any time τ , $\chi(\tau)$ is the GDP. The GDP at initial time $\tau = 0$ is χ_0 .

Solution: Taking Elzaki transform to Eq. (6), we have

$$\begin{aligned} \chi(t) - t^2 \chi(0) &= \lambda t \chi(t), \\ (1 - \lambda t) \chi(t) &= t^2 \chi(0) \end{aligned}$$

We put $\chi(0) = \chi_0$ and rewriting, we have

$$\chi(t) = \frac{t^2 \chi_0}{(1 - \lambda t)}$$

Taking inverse Elzaki transform, we have
 $\chi(\tau) = \chi_0 e^{\lambda\tau}$.

$\lim_{\tau \rightarrow \infty}$, τ approaches infinity, $\chi = 0$ for negative value of λ , $\chi = \infty$ for positive value of λ .

From the above equation, it is clear that GDP falls off and shoots up exponentially.

2.5. Free fall in a Viscous Medium: Under the operation of gravity pull in a viscous medium, the motion of a particle falls vertically downwards

$$\frac{d\vartheta(\tau)}{d\tau} = -\lambda\vartheta(\tau) + G \quad (7)$$

Here $-\lambda\vartheta(\tau)$ is due to the force of air resistance and G is the acceleration due to gravity. Also, we assume $\vartheta(0) = 0$.

Solution: Taking Elzaki transform to Eq. (7), we have

$$\vartheta(t) - t^2\vartheta(0) = -\lambda t\vartheta(t) + G t^3$$

We put $\vartheta(0) = 0$ and rewriting, we have

$$(1 + \lambda t)\vartheta(t) = G t^3,$$

$$\vartheta(t) = \frac{G}{\lambda} \left(t^2 - \frac{t^2}{(1 + \lambda t)} \right),$$

$$\vartheta(t) = -\frac{G}{\lambda} \frac{t^2}{(1 + \lambda t)} + \frac{G}{\lambda} t^2,$$

$$\vartheta(t) = -\frac{G}{\lambda} \frac{t^2}{(1 + \lambda t)} + \frac{G}{\lambda} t^2,$$

Taking inverse Elzaki transform, we have

$$\vartheta(\tau) = -\frac{G}{\lambda} e^{-\lambda\tau} + \frac{G}{\lambda},$$

$$\vartheta(\tau) = \frac{G}{\lambda} (1 - e^{-\lambda\tau}).$$

$\lim_{\tau \rightarrow \infty}$, τ approaches infinity, then $\vartheta = \frac{G}{\lambda}$ which show maximum velocity of the particle falling vertical under gravity.

2.6. Law of heating or cooling

The law of heating or cooling is given by Newton which is described as [8]- [12]

$$\frac{d\mathcal{T}(\tau)}{d\tau} = -\lambda(\mathcal{T}(\tau) - \mathcal{T}_0), \quad (8)$$

in surrounding medium, the fixed temperature is $\mathcal{T}_0$. and at any instant $\mathcal{T}$, the temperature is $\mathcal{T}$. So we assume that

$$\mathcal{T}(0) = \mathcal{T}_0.$$

Solution: Taking the Elzaki transform to Eq. (8)

$$\mathcal{T}(t) - t^2\mathcal{T}(0) = -\lambda(t\mathcal{T}(t) - \mathcal{T}_0 t^3)$$

We put $\mathcal{T}(0) = \mathcal{T}_1$ and solving, we have

$$(1 + \lambda t)\mathcal{T}(t) = t^2\mathcal{T}(0) + \lambda\mathcal{T}_0 t^3,$$

$$\mathcal{T}(t) = \frac{t^2 \mathcal{T}_1}{(1+\lambda t)} + \frac{\lambda \mathcal{T}_0 t^3}{(1+\lambda t)},$$

$$\mathcal{T}(t) = \frac{t^2 (\mathcal{T}_1 - \mathcal{T}_0)}{(1+\lambda t)} + \mathcal{T}_0 t^2,$$

taking the inverse Elzaki transform, we have

$$\mathcal{T}(\tau) = (\mathcal{T}_1 - \mathcal{T}_0) e^{-\lambda \tau} + \mathcal{T}_0.$$

From the above equation, it is clear that the temperature of the body falls off exponentially.

3. Conclusion: The Elzaki transform is highlighted as an effective mathematical tool for solving differential equations in various scientific contexts, offering solutions that align with those obtained via traditional calculus methods. Here's an outline of its applications such as Population growth or decline is modeled exponentially. Population approaches zero if the death rate exceeds the birth rate. Population approaches infinity if the birth rate exceeds the death rate. The concentration of glucose in the bloodstream decreases exponentially. Over a long period, all susceptible individuals eventually become infected. GDP can increase or decrease exponentially over time. The temperature of an object changes exponentially over time. The velocity of an object falling in a viscous medium increases exponentially, with g/λ being the terminal velocity. This validates the Elzaki transform

as a powerful tool for addressing problems in physical and basic sciences.

Availability of data and materials: All data supporting the findings of this study are included in the article and the cited references.

Conflict of Interest: The authors declare that there is no conflict of interest regarding this study.

Author's Contribution: The study was directed by Ritu Jain, who also carried out all mathematical computations. Manvendra Narayan Mishra, constructed the study map. Nikita Kumawat formatted the final manuscript. The draft was read, corrected, and polished by the author.

Funding: This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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