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A SOLUTION OF PARTIAL DIFFERENTIAL EQUATION ASSOCIATED WITH I- FUNCTION AND GENERALIZED M-SERIES IN THE STUDY OF ANGULAR DISPLACEMENT

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Abstract: In an attempt to give extension of the result in the theory of special functions, we discuss the application of certain products involving I- function [10] and a generalized M-series [12] in obtaining a solution of the partial differential equation $\frac{\partial^2 f}{\partial t^2} = R^2 \frac{\partial^2 f}{\partial y^2}$ Concerning to a problem of angular displacement in a shaft.

Key-words: I-function, partial differential equation, generalized M-series, angular displacement.

INTRODUCTION

The I- function given by Saxena [10] is represented and defined as following:

$$I^{m,n}_{p_i,q_i;r}[Z] = I^{m,n}_{p_i,q_i;r}\left[z \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right]$$

$$= \frac{1}{2\pi i}\int \phi(\xi)\, z^{\xi}\, d\xi \qquad \dots (1.1)$$

Where

$$\phi(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j\xi) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j\xi)}{\sum_{i=1}^{r} c_i \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji}\xi) \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji}\xi)} \qquad \dots (1.2)$$

p_i (i= 1,….,r), q_i (i= 1,….,r), m, n are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i$ (i= 1,….,r); r is finite, α_j, β_j, α_{ji}, β_{ji}, are real and positive; a_j, b_j, a_{ji}, b_{ji} are complex numbers and £ is the path of integration separating the increasing and decreasing sequences of poles of the integrand.

The integral converges if

$$|\arg(z)| < \frac{\pi}{2}\Omega_i ,$$

Where

$$\Omega_i = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_j > 0$$

And

$$T = \sum_{j=1}^{q_i} b_j - \sum_{j=1}^{p_i} a_j > 0 \qquad \dots (1.3)$$

If we take r=1 in (1.1), then the I- function will convert to the well known Fox's H- function.

The generalized M-series is the extension of the both Mitag- Laffler function and generalized hyper geometric function.

It is represented as following:

$$\overset{\alpha,\beta}{\underset{p\ q}{M}}(u_1,\dots,u_p;v_1,\dots,v_q;y) = \overset{\alpha,\beta}{\underset{p\ q}{M}}(y) = \sum_{K=0}^{\infty} \frac{(u_1)_k \dots\dots (u_p)_k}{(v_1)_k \dots\dots (v_q)_k} \frac{y^k}{\Gamma(\alpha k + \beta)}$$

$y, \alpha, \beta \in C$, Re $(\alpha) > 0$ … (1.4)

Here $(u_j)_k$,$(v_j)_k$ are the known pochammer symbols. The series (1.4) is defined when none of the parameters v_j's (j= 1, 2,…, q), is a negative integer or zero. If any numerator parameter u_j is a negative integer or zero then the series terminates to a polynomial in z. The series (1.4) is convergent for all y if p ≤ q.

We consider the problem of determining the twist f(y,t) in a shaft of circular section with its axis along the y-axis. Now the displacement f(y,t) due to initial twist must satisfy the boundary value problem. If we assume that both the ends y = 0 and y = υ of the shaft are free

$$\frac{\partial^2 f}{\partial t^2} = \mathrm{R}^2 \frac{\partial^2 f}{\partial y^2} \qquad \ldots (1.5)$$

Where R is a constant

$$\frac{\partial f}{\partial y}(0,t) = 0, \frac{\partial f}{\partial y}(y,0) \text{ and } f(y,0) = \psi(y) \qquad \ldots (1.6)$$

$$\text{Let} \quad \psi(y) = \left(\sin \frac{\pi y}{2\upsilon}\right)^{2\delta-\mu-1} \left(\cos \frac{\pi y}{2\upsilon}\right)^{\mu-1} \underset{p\ \ q}{\overset{\alpha\ ,\ \beta}{M}} \left[x\left(\tan \frac{\pi y}{2\upsilon}\right)^{2\lambda} \right] . \, I^{m,n}_{p_i,q_i;r} \left[z\left(\tan \frac{\pi y}{2\upsilon}\right)^{2l} \right] \qquad \ldots (1.7)$$

1. **The Integral Relation:**

Here we establish the following integral relation:

$$\int_0^{\upsilon} \left(\cos \frac{\pi \delta y}{\upsilon}\right) \left(\sin \frac{\pi y}{2\upsilon}\right)^{2\delta-\mu-1} \left(\cos \frac{\pi y}{2\upsilon}\right)^{\mu-1} \underset{p\ \ q}{\overset{\alpha\ ,\ \beta}{M}} \left[x\left(\tan \frac{\pi y}{2\upsilon}\right)^{2\lambda} \right] I^{m,n}_{p_i,q_i;r} \left[z\left(\tan \frac{\pi y}{2\upsilon}\right)^{2\lambda} \right] dy$$

$$= \frac{\upsilon 2^{2\delta-\mu+2\lambda k}}{\Gamma(2\delta)\sqrt{\pi}} \sum_{K=0}^{\infty} \frac{(u_1)_k \ldots\ldots (u_p)_k}{(v_1)_k \ldots\ldots (v_q)_k} \frac{x^k}{\Gamma(\alpha k + \beta)}$$

$$I^{m+1,n+1}_{p_i+2,q_i+1,\ ;r} \left[\frac{1}{z\,4^{\lambda}} \middle| \begin{matrix} \left(1-\delta+\frac{\mu}{2}-\lambda k,\lambda;1\right),(a_j,A_j)_{1,n},[(a_{ji},A_{ji})]_{n+1,p_i;r},(\mu-\lambda k;2\lambda) \\ \left(\frac{1}{2}-\delta+\frac{\mu}{2}-\lambda k;\lambda\right),(b_j,B_j)_{1,m},[C_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right]$$

$$\ldots (2.1)$$

Where $\lambda > 0, \lambda > 0, \mathrm{Re}\left(\mu - 2\lambda \frac{b_j}{B_j}\right) > 0 (j = 1,\ldots,m),$ m is an arbitrary positive integer and α, β ∈ C.

Proof: The integral in (2.1) can be established by using the definition of generalizes M-series given by (1.4) and I-function in terms of Mellin-Barnes contour integral given by (1.1), then interchanging the order of summation and integration, obtain the inner integral with the help of a result given by Chaurasia and Gupta [2], and we reach at the desired result.

2. **Solution of the Problem posed**: The solution of the problem to be established is:

$$f(y,t) = \frac{1}{2^{\mu}\sqrt{\pi}} \sum_{K=0}^{\infty} \frac{(u_1)_k \ldots\ldots (u_p)_k}{(v_1)_k \ldots\ldots (v_q)_k} \frac{x^k}{\Gamma(\alpha k + \beta)} \frac{2^{2\tau+2\lambda k}}{\Gamma(2\tau)}$$

$$I^{m+1,n+1}_{p_i+2,q_i+1,\ ;r} \left[\frac{1}{z\,4^{\lambda}} \middle| \begin{matrix} \left(1-\tau+\frac{\mu}{2}-\lambda k,\lambda;1\right),(a_j,A_j)_{1,n},[(a_{ji},A_{ji})]_{n+1,p_i;r},(\mu-\lambda k;2\lambda) \\ \left(\frac{1}{2}-\tau+\frac{\mu}{2}-\lambda k;\lambda\right),(b_j,B_j)_{1,m},[C_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right]$$

$$\left(\cos \frac{\pi \tau y}{\upsilon}\right)\left(\cos \frac{\pi \tau \mathrm{R}\, t}{\upsilon}\right) \qquad \ldots (3.1)$$

This is valid under the same conditions required for (2.1)

3. **Derivation of (3.1)**: The solution of the problem can be written as ([4], Churchill, 1941, p.125 (4)).

$$f(y,t) = \frac{1}{2}a_0 + \sum_{\tau=1}^{\infty} a_{\tau} \left(\cos \frac{\pi y \tau}{\upsilon}\right)\left(\cos \frac{\pi \tau \mathrm{R} t}{\upsilon}\right) \qquad \ldots (4.1)$$

Where $a_{\tau} (\tau = 0,1,2,\ldots)$ are the coefficients in the Fourier Cosine Series for ψ(y) in the interval (0, υ).

If t = 0, then by virtue of (1.7), we get

$$\left(\sin \frac{\pi y}{2\upsilon}\right)^{2\delta-\mu-1} \left(\cos \frac{\pi y}{2\upsilon}\right)^{\mu-1} \underset{p\ \ q}{\overset{\alpha\ ,\ \beta}{M}} \left[x\left(\tan \frac{\pi y}{2\upsilon}\right)^{2\lambda} \right] .$$

$$I^{m,n}_{p_i,q_i,c_i;r} \left[z\left(\tan \frac{\pi y}{2\upsilon}\right)^{2\lambda} \right] = \frac{1}{2}a_0 + \sum_{\tau=1}^{\infty} a_{\tau} \left(\cos \frac{\pi \tau y}{\upsilon}\right) \qquad \ldots (4.2)$$

Now multiplying both sides of (4.2) by $\left(\cos \frac{\pi \delta y}{\upsilon}\right)$ and integrating with respect to y from 0 to υ, we get

$$\int_0^{\upsilon} \left(\cos \frac{\pi \delta y}{\upsilon}\right) \left(\sin \frac{\pi y}{2\upsilon}\right)^{2\delta-\mu-1} \left(\cos \frac{\pi y}{2\upsilon}\right)^{\mu-1} \underset{p\ \ q}{\overset{\alpha\ ,\ \beta}{M}} \left[x\left(\tan \frac{\pi y}{2\upsilon}\right)^{2\lambda} \right] I^{m,n}_{p_i,q_i;r} \left[z\left(\tan \frac{\pi y}{2\upsilon}\right)^{2\lambda} \right] dy$$

$$= \frac{1}{2}a_0 \int_0^{\upsilon} \left(\cos \frac{\pi \delta y}{\upsilon}\right) dy + \sum_{\tau=1}^{\infty} a_{\tau} \left(\cos \frac{\pi \tau y}{\upsilon}\right)\left(\cos \frac{\pi \delta y}{\upsilon}\right) dy \qquad \ldots (4.3)$$

Now by using (2.1) along with orthogonal property of the cosine functions, we get

$$a_{\tau} = \frac{2^{2\tau-\mu+2\lambda k}}{\Gamma(2\tau)\sqrt{\pi}} \sum_{K=0}^{\infty} \frac{(u_1)_k \ldots\ldots (u_p)_k}{(v_1)_k \ldots\ldots (v_q)_k} \frac{x^k}{\Gamma(\alpha k + \beta)}$$

$$I^{m+1,n+1}_{p_i+2,q_i+1,\ ;r}\left[\frac{1}{z\,4^{\lambda}}\left|\begin{matrix}\left(1-\tau+\frac{\mu}{2}-\lambda k,\lambda;1\right),(a_j,A_j)_{1,n},[(a_{ji},A_{ji})]_{n+1,p_i;r},(\mu-\lambda k;2\lambda)\\ \left(\frac{1}{2}-\tau+\frac{\mu}{2}-\lambda k;\lambda\right),(b_j,B_j)_{1,m},[C_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right] \quad \ldots (4.4)$$

Now by using (4.1) and (4.4), we get the desired solution in (3.1).

4. Particular cases:

(i)

$$\int_0^{\upsilon}\left(\cos\frac{\pi\delta y}{\upsilon}\right)\left(\sin\frac{\pi y}{2\upsilon}\right)^{2\delta-\mu-1}\left(\cos\frac{\pi y}{2\upsilon}\right)^{\mu-1}$$

$$\left[x\left(\tan\frac{\pi y}{2\upsilon}\right)^{2\lambda}\right] S^{m_1,\ldots,m_s}_{n_1,\ldots,n_s}\left[x_1\left(\tan\frac{\pi y}{2\upsilon}\right)^{2\lambda_1},\ldots,\right.$$

$$\left.x_s\left(\tan\frac{\pi y}{2\upsilon}\right)^{2\lambda_s}\right] I^{m,n}_{p_i,q_i;r}\left[z\left(\tan\frac{\pi y}{2\upsilon}\right)^{2\lambda}\right]dy$$

$$= \frac{\upsilon.2^{2\delta-\mu+2(\lambda k+\sum_{i=1}^{s}\lambda_i k_i)}}{\Gamma(2\delta)\sqrt{\pi}} \sum_{K=0}^{\infty}$$

$$\frac{(u_1)_k \ldots\ldots (u_p)_k}{(v_1)_k \ldots\ldots (v_q)_k} \frac{x^k}{\Gamma(\alpha k + \beta)}$$

$$\sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1}}{k_1!} \cdots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1,k_1;\ldots;n_s,k_s]\, x_1^{k_1}\ldots x_s^{k_s}$$

$$I^{m+1,n+1}_{p_i+2,q_i+1,\ ;r}\left[\frac{1}{z\,4^{\lambda}}\left|\begin{matrix}\left(1-\delta+\frac{\mu}{2}-\lambda k-\sum_{i=1}^{s}\lambda_i k_i,\lambda;1\right),(a_j,A_j)_{1,n},[(a_{ji},A_{ji})]_{n+1,p_i;r},\left(\mu-\lambda k-\sum_{i=1}^{s}\lambda_i k_i;2\lambda\right)\\ \left(\frac{1}{2}-\delta+\frac{\mu}{2}-\lambda k-\sum_{i=1}^{s}\lambda_i k_i;\lambda\right),(b_j,B_j)_{1,m},[C_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]$$

$$\ldots (5.1)$$

Valid under the conditions which are true for (2.1) and (5.1)

(ii) Taking generalized polynomials [14] in place of M-series in (2.1), we get

$$\int_0^{\upsilon}\left(\cos\frac{\pi\delta y}{\upsilon}\right)\left(\sin\frac{\pi y}{2\upsilon}\right)^{2\delta-\mu-1}\left(\cos\frac{\pi y}{2\upsilon}\right)^{\mu-1}$$

$$S^{m_1,\ldots,m_s}_{n_1,\ldots,n_s}\left[x_1\left(\tan\frac{\pi y}{2\upsilon}\right)^{2\lambda_1},\ldots,x_s\left(\tan\frac{\pi y}{2\upsilon}\right)^{2\lambda_s}\right]$$

$$.\,I^{m,n}_{p_i,q_i;r}\left[z\left(\tan\frac{\pi y}{2\upsilon}\right)^{2\lambda}\right]dy$$

$$= \frac{\upsilon 2^{2\delta-\mu+2\sum_{i=1}^{s}\lambda_i k_i}}{\Gamma(2\delta)\sqrt{\pi}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1}}{k_1!} \cdots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1,k_1;\ldots;n_s,k_s]\, x_1^{k_1}\ldots x_s^{k_s}$$

$$I^{m+1,n+1}_{p_i+2,q_i+1,\ ;r}\left[\frac{1}{z\,4^{\lambda}}\left|\begin{matrix}\left(1-\delta+\frac{\mu}{2}-\sum_{i=1}^{s}\lambda_i k_i,\lambda;1\right),(a_j,A_j)_{1,n},[(a_{ji},A_{ji})]_{n+1,p_i;r},\left(\mu-\sum_{i=1}^{s}\lambda_i k_i;2\lambda\right)\\ \left(\frac{1}{2}-\delta+\frac{\mu}{2}-\sum_{i=1}^{s}\lambda_i k_i;\lambda\right),(b_j,B_j)_{1,m},[C_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]$$

$$\ldots (5.2)$$

Where $k_i > 0 (i=1,\ldots,s), h > 0,\ \lambda \phi 0 \operatorname{Re}\left(\mu - 2\lambda\frac{b_j}{B_j}\right) > 0 (j=1,\ldots,m),$ m is an arbitrary positive integer and the coefficient $A[n_1,k_1;\ldots;n_s,k_s]$ are arbitrary constants, real or complex.

Now taking r = 1 and s = 2 and $\lambda_i \to 0$ in (5.1), we get the known result obtained by Chaurasia and Godika [1].

(ii) Taking $i \to 1$ and $r \to 1$ in (5.1), we get the known result obtained by Chaurasia and Shekhawat [3]

(iii) Taking Aleph function in place of I- function in (5.1), we get the known result obtained by Shekhawat and Garg (13).

CONCLUSION

In this paper, the established result is very useful in many interesting situations appearing in the literature on mathematical analysis, applied mathematics and mathematical physics with the help of our result. We found the angular displacement in a shaft –III by using special function (I- function).

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